UNIVERSITÀ DI PARMA

UNIVERSITÀ DEGLI STUDI DI PARMA

DOTTORATO DI RICERCA IN
MATEMATICA

Ciclo XXXV

Formality of special complex manifolds: deformations and cohomological properties

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Anni Accademici 2019–2022

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Introduction

On a compact complex manifold, the existence of special metric structures can yield many information concerning its topological and algebraic invariants. As a foremost example, the property of admitting a Kähler metric, i.e., a Hermitian metric whose associated fundamental form is closed, imposes strong restraints on a complex manifold: from the topological point of view, the *Betti numbers* b_k, i.e., the dimensions of the k-th de Rham cohomology spaces of the manifold, must satisfy

$$\begin{cases} b_k > 0, & k \equiv 0 \mod 2 \\ b_k \equiv 0, & k \equiv 1 \mod 2. \end{cases}$$

Moreover, compact complex manifolds admitting a Kähler metric, shortly *Kähler manifolds*, satisfy the *Kähler identities* (see, e.g., [75, Section 3.1]), which yield that the Dolbeault Laplacian and its conjugate are multiples of the usual Hodge Laplacian, hence the harmonic representatives of the de Rham cohomology and Dolbeault cohomology coincide. Combined with Hodge theory, this assures that compact Kähler manifolds satisfy the *$\partial\overline{\partial}$-lemma*, i.e., the very special property that every d^c-closed d-exact form on the manifold is also $\partial\overline{\partial}$-exact (see [46]), where, as usual, $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)$. This property forces the natural complex cohomologies associated to a complex manifold, that is, Dolbeault, Bott-Chern, and Aeppli cohomologies (see section 1.3 for the definitions), to be isomorphic; it turns out that on a Kähler manifold they indeed coincide. Hence, on a compact Kähler manifold, the Hodge decomposition holds and the Frölicher spectral sequence degenerates at the first step (see, e.g., [64]). From the algebraic point of view, Kähler manifolds are formal according to Sullivan and every Massey product vanishes; as a consequence the homotopy type is a "formal consequence" of its cohomology ring, see [143]. Examples of compact complex manifolds admitting a Kähler metric are complex tori $\mathbb{T}^n := \Gamma\backslash\mathbb{C}^n$, i.e., compact quotients of the complex space $\mathbb{C}^n$ by a discrete uniform subgroup Γ, as they inherit a Kähler metric from the standard Kähler metric $g = \frac{i}{2}\sum_{i=1}^n dz^i \wedge d\overline{z}^i$ on $\mathbb{C}^n$. Furthermore, there exist suitable tools to construct a Kähler manifold starting from a known Kähler manifold: by a celebrated result of Kodaira and Spencer in [86], the property of admitting a Kähler metric is open under small deformations of the complex structure, i.e., any infinitesimal deformation of a Kähler manifold is still a Kähler manifold, and either restricting a Kähler metric to a complex submanifold or performing a complex blow-up of a Kähler manifold along a complex submanifold (see [154]) yields a Kähler manifold.

However, because of the many restrictions that admitting a Kähler metric imposes, a complex manifold is not Kähler in general, e.g., the only Kähler manifolds in the class of *nilmanifolds*, i.e., compact quotients of connected, simply connected nilpotent Lie groups by a discrete uniform subgroup, are tori, see [24, Theorem A], and more in general, a *solvmanifold*, i.e, a compact quotient of a simply-connected solvable Lie group by a closed subgroup, carries a Kähler structure if and only if it is a finite quotient of a complex torus, by Hasegawa (see [67] and also [68])). As a consequence, starting from the 80's a number of authors have introduced notions of new Hermitian metric structures which generalize Kähler metric. This has been accomplished by defining Hermitian metrics whose associated fundamental forms belong to the kernels of differential operators associated to the complex structure of the complex manifolds, thus yielding interesting Hermitian metrics, e.g., *strong Kähler with torsion* metrics, *astheno-Kähler* metrics, and *balanced metrics*. Each of the mentioned notions arises naturally in relevant settings of complex geometry, and, in particular for strong Kähler with torsion metrics, also in theoretical physics (see, e.g., [59, 141, 77]). More in details, let (M, J) be a n-dimensional complex manifold and let g be a Hermitian metric on (M, J) and $\omega(X, Y) := g(JX, Y)$, $X, Y \in TM$ its associated fundamental form.

Strong Kähler with torsion metrics ([26]). The metric g is said to be *strong Kähler with torsion*, shortly *SKT*, if

$$\partial\overline{\partial}\omega = 0.$$

Strong Kähler with torsion metrics arise in the context of *Kähler with torsion geometry*. More precisely, given a Hermitian manifold (M, J, g, ω), in [26] Bismut showed that there exists an unique connection ∇^B, known as *Bismut connection*, which preserves the Hermtian metric g and the complex structure J, i.e., $\nabla^B g = 0$, $\nabla^B J = 0$, and for which the tensor $g(X, T(Y, Z))$ is totally skew-symmetric, where T stands for the torsion of the connection ∇^B. The properties of such connection are related to what is called *Kähler with torsion geometry* (we refer to [51, 60, 114, 77, 141] for further details). The tensor $g(\cdot, T(\cdot, \cdot))$ can then be identified with the 3-form $Jd\omega$ and, in the particular case in which this form is closed or, equivalently, $\partial\overline{\partial}\omega = 0$, we say that the Hermitian metric g is *strong Kähler with torsion*, shortly, *SKT*. Such metrics have relevant relations with generalized Kähler geometry (see for instance [59, 65, 74, 21, 39, 38, 145]) and type II string theory and in 2-dimensional supersymmetric σ-models (see, e.g., [59, 141, 77]). For compact complex surfaces, i.e., 2-dimensional compact complex manifolds, the notion of strong Kähler with torsion metrics is equivalent to the notion of regular metrics according to Gauduchon, see [62]; hence, by the celebrated Gauduchon theorem [62, Theorem 1], on a compact complex surface, there exists a unique SKT metric in every equivalence class of conformal Hermitian metrics. In higher dimension, this does not hold, e.g., in complex dimension 3 Fino, Parton, and Salamon have classified real 6-dimensional nilmanifolds endowed with invariant complex structures admitting a SKT metric, see [53, Theorem 1.2]. In particular, the authors show that on any such manifold either every invariant Hermitian metric is SKT or none is, and every invariant Hermitian metric is SKT if, and only if, the structure constants associated to the complex structure of the manifold satisfy a certain relation (see (6.4.1)). Existence results have been proved also in complex dimension 4 by Rossi and Tomassini in [125], in which the authors show that, in contrast to complex dimension 3, there exist invariant complex structures admitting both SKT and non SKT invariant metrics and they provide sufficient conditions on the structure constant of any 8-dimensional nilmanifold endowed with a left-invariant complex structure under which any invariant Hermitian metric is SKT. We point out that compact complex manifolds admitting SKT structures have been proven to be valid candidates for the study of generalizations of the Kähler-Ricci flow, see for example [139].

Astheno-Kähler metrics ([78]). The metric g on a (M, J) is said to be *astheno-Kähler* if

$$\partial\overline{\partial}\omega^{n-2} = 0.$$

Astheno-Kähler metrics have been introduced by Jost and Yau in [78] in the study of systems of elliptic non linear equations related to rigidity results for complex manifolds. In particular, the authors have shown that if a compact complex manifold (M, J) admitting an astheno-Kähler metric is homotopy equivalent to a non compact locally Hermitian symmetric spaces without the upper plane as a factor of the universal cover or, respectively, is homotopy equivalent to a compact Kähler manifold with additional assumptions on the curvature, then (M, J) is biholomorphic to said locally Hermitian symmetric space, respectively, Kähler manifold. By dimension arguments, every Hermitian metric on a compact complex surface is astheno-Kähler. In complex dimension 3, the notion of astheno-Kähler metrics coincides with the notion of strong Kähler with torsion metrics; therefore, in the setting of nilmanifolds admitting invariant complex structures, Fino-Parton-Salamon's results apply. However, already from complex dimension 4, the two notions are independent, e.g., in [125] Rossi and Tomassini study the relation between astheno-Kähler and strong Kähler with torsion metrics on nilmanifolds real dimension 8 endowed with a invariant complex structure. In fact, they provide sufficient conditions on the structure constants of the complex structure under which any invariant Hermitian metric is astheno-Kähler, respectively strong Kähler with torsion, and they construct examples of manifolds admitting both astheno-Kähler metrics and non astheno-Kähler metrics. Furthermore, astheno-Kähler structures on Calabi-Eckmann manifolds have been constructed in [100].

Balanced metrics ([103]). A Hermitian metric g on (M, J) is said *balanced*, or *co-Kähler*, if

$$d\omega^{n-1} = 0.$$

Balanced metrics have been introduced by Michelson in [103], as a class of Hermitian metrics which generalizes the notion of Kähler metrics by requiring weaker assumptions on the torsion tensor of the *Chern connection*. In particular, on a n-dimensional Hermitian manifold (M, J, g, ω) the Chern connection is a connection ∇^{Ch} which preserves the complex structure J and the Hermitian metric g, and whose $(1,1)$-component of the torsion T^{Ch} is identically zero. Then, the metric g is Kähler if, and only if, T^{Ch} is identically zero. Michelson defines balanced metrics as Hermitian metrics such that the 1-form obtained by taking the trace of T^{Ch} vanishes, i.e., $\tau := \mathrm{tr}(T^{Ch}) \equiv 0$. As it turns out, this condition is equivalent to the non linear equation $d\omega^{n-1} = 0$, or, equivalently, $d^*\omega = 0$ (hence, the term *co-Kähler*), where d^* is the formal adjoint to d with respect to the L^2 scalar product on differential forms induced by g (see section 1.3). Balanced manifolds are in some sense dual to Kähler manifolds, e.g., if $f\colon X \to Y$ is a holomorphic immersion from a complex manifold into a Kähler manifold, then X is also Kähler, whereas if $g\colon X \to Y$ is a holomorphic submersions of a balanced manifold onto a complex manifold, then Y is balanced. Regarding the existence of balanced metrics on compact complex manifolds, in complex dimension 2 this metric notion coincides with the notion of Kähler metrics, whereas for complex dimension $n \geq 3$, many examples of balanced non-Kähler can be constructed as total space of family of Kähler manifolds parametrized over a complex line, see [103, Theorem 5.5]. Moreover, in [6, Remark 3.1], Alessandrini and Bassanelli have showed that any compact holomorphically parallelizable manifold is balanced and Ugarte proved a classification of balanced structures on 6-dimensional nilmanifolds endowed with nilpotent complex structures in [152]. Furthermore, whereas for SKT metrics it has been conjectured that on the same non-Kähler compact complex manifold there cannot exist both SKT metrics and balanced metrics with respect to the same complex structure (see [56]), in [101] it has been indeed proved that an astheno-Kähler metric is balanced if and only if the metric is also Kähler, but in [52] the authors show the existence of a compact complex non-Kähler manifold which admits both a balanced and astheno-Kähler metric.

In the wake of the results by Harvey and Lawson on Kähler manifolds in [66], for a compact complex manifold (M, J) the property of admitting any of the above metrics has been characterized in terms of currents by, respectively, Michelson [103, Theorem 4.7], Alessandrini [2, Theorem 2.4], and Egidi [48, Theorem 3.3]. In particular, the fundamental forms of strong Kähler with torsion metrics and astheno-Kähler metrics belong to the family of *p-pluriclosed forms* for $p = 1$, respectively, for $p = n - 2$, and the fundamental forms of Kähler metrics and balanced metrics belong to the family of *p-Kähler forms* for $p = 1$, respectively, $p = n - 1$, where $n = \dim_{\mathbb{C}}(M, J)$. For the precise definitions, see section 1.5.

Geometrically formal metrics. Alongside the above metrics, in this work a relevant role is played by the study of classes of metrics that arise in homotopic theory as introduced by Sullivan in [143]. More precisely, a complex manifold (M, J) is said to *formal according to Sullivan*, if its de Rham complex $(\mathcal{A}^\bullet_{\mathbb{C}}(M), d)$ is equivalent, in the category of differential graded algebras, the algebra of its de Rham cohomology $(H^\bullet_{dR}(M), 0)$; in this situation, the complex of differential forms and the complex of its de Rham cohomology share the same minimal model and, hence, the homotopic type of the manifold is a "formal consequence" of this cohomology ring. Compact Kähler manifolds and, more in general, compact manifolds satisying the $\partial\overline{\partial}$-lemma, are formal according to Sullivan. An obstruction to this property is represented by the presence of non vanishing *Massey triple products*, introduced in [99], i.e., elements of quotients of the de Rham cohomology by an indeterminacy ideal (see Appendix B). Concerning the de Rham cohomology of a differentiable manifold, it is not possible, a priori, to fix canonical representatives so that they are an algebra with respect to the $\wedge$ product; however, as Sullivan noticed in [144], if (M, J) admits a Hermitian metrics such that the space of harmonic forms has a structure of algebra induced by the $\wedge$ product, then the manifold (M, J) is also formal according to Sullivan. The metrics for which the above property holds are called *geometrically formal (according to Kotschick)*, see [87], and also the existence of such metrics is obstructed by of the existence of non vanishing Massey triple products. In order to study a notion of "holomorphic homotopy theory", Neisendorfer and Taylor have introduced in [106] the notion of *Dolbeault formality* and *Dolbeault Massey triple products* (see Section 1.4) as natural adaptations of Sullivan's formality involving the holomorphic structure of a complex manifold. Note that Dolbeault formality implies that every Dolbeault Massey triple product vanishes and compact complex manifolds satisfying the $\partial\overline{\partial}$-lemma are Dolbeault formal (see [106, Theorem 8, Section 7]). In relation to this, Tomassini and Torelli in [150], respectively, Angella and Tomassini in [18], have then introduced the notions of *geometrically-Bott-Chern-formal metrics*, and *geometrically-Bott-Chern-formal metrics* and *Aeppli-Bott-Chern-Massey triple products*, which we are reviewed at the end of Section 1.4. The links between Dolbeault formality, Dolbeault-geometrically-formal metrics, and Dolbeault-Massey triple products are clear (see (1.4.2) and (1.4.3)), whereas only recently Stelzig and Milivojevic have introduced in [104] a notion of formality which can be interpreted as "Bott-Chern formality" and which is related to both geometrically-Bott-Chern-formal metrics and Aeppli-Bott-Chern-Massey triple products.

The aim of this thesis work is to study the deformation and cohomological properties of such special metric structures in a general setting and on concrete examples of both classical and more recently introduced families of compact complex manifolds. Our goal is also to outline the interplay between geometrically formal metrics and generalizations of Kähler metrics. Such study can indeed lead to new interesting tools in a more deep understanding of complex manifolds.

In particular this work is divided as follows.

In the first chapter, we recall the main facts that will be needed through the work. More specifically, complex manifolds, their cohomologies and Hodge theory, the complex formalities and the special metric structures, the tools to computate the complex cohomologies of complex manifolds with a structure of compact quotient of Lie group, and a brief review of deformation theory.

In the second chapter, we study deformations of the notions of the above special metrics. In fact, whereas the Kähler condition is stable under the action of deformations of the complex structure of a compact complex manifold, i.e., any small deformations of a Kähler manifold still admit a Kähler metric, it has been shown in [8], respectively [55], that the same does not hold for balanced, respectively, SKT and AK metrics, by providing examples of deformations of balanced, respectively strong Kähler with torsion and astheno-Kähler metrics, which do not admit any of the respective metrics. Therefore, it seemed natural to ask whether there exist cohomological obstructions to the construction of curves of special metrics along curve of deformations, where by a "curve of special metrics along a curve of deformations" we mean a 1-parameter family of special metrics $\{\omega_t\}_t$ along a 1-parameter family of deformations $\{(M, J_t)\}_t$ of a compact manifold (M, J) such that, if $(M, J_{t_0}) = (M, J)$, then ω_{t_0} coincides with special metric ω on (M, J). This approach led to the necessary conditions described in Theorems 2.2.1, 2.3.1, 2.4.1 and their following Corollaries. These results are then a useful tool to obtain obstructions to the existence of deformations by curve of special metrics on explicit examples of manifolds. For strong Kähler with torsion metrics and astheno Kähler metrics, we study such obstructions on two different families of nilmanifolds of complex dimension 4 introduced first contructed in [55] (see also [125]), whereas for balanced metrics, we study two examples of non tori holomorphically parallelizable solvmanifolds in complex dimension 3, i.e, the *Iwasawa manifold* and the *holomorphically parallelizable Nakamura manifold*.

In the third chapter of this work, we study the existence of p-Kähler structures (in particular, balanced metrics) and SKT metrics on a family of compact complex manifolds of complex dimension $4n-2$ with $n \geq 2$, introduced by Bigalke and Rollenske in [25] to prove that the degeneration step of the Frölicher spectral sequence can be arbitrarily large. More precisely, we preliminarly prove obstructions to the existence of p-Kähler structures on nilmanifolds with nilpotent left-invariant complex structures (see Theorem 3.1.2). With the aid of such obstructions, we are able to prove that the Bigalke-Rollenske manifolds there exists no p-Kähler structure for $p \in \{2, \dots, 4n-3\}$ but the canonical diagonal metric is balanced (Theorems 3.2.2 and 3.2.3); hence, this proves that, in contrast to Kähler manifolds, there exists no correlation between the degeneracy step of the Frölicher spectral sequence and the property of admitting a balanced metric. Furthermore, any element of the Bigalke and Rollenske manifolds does not admit SKT metrics (Proposition 3.2.5) nor locally conformally Kähler metrics (Proposition 3.2.6, by combining results in [110]).

In the fourth chapter of this thesis, we study the behaviour under deformations of the notions of complex formality as previously recalled. In fact, it has been shown by Tomassini and Torelli [150, Theorems 4.1, 4.2, 4.3] that Dolbeault formality, geometric Dolbeault formality, and the property of admitting non vanishing Dolbeault Massey products are not stable under deformations. Analogously, Tardini and Tomassini have shown that geometric Bott-Chern formality and the property of admitting non vanishing Aeppli-Bott-Chern Massey products are not stable under deformation of the complex structure (see [147, Corollary 4.5]). Hence, it seemed interesting to check whether the above properties satisfy any other stability property under deformations. As a result, by constructing two explicit examples, we are able to prove that the neither the "Dolbeault formalities" neither the "Bott-Chern formalities" are closed under deformations (in sense of Definition 1.7.3), see Theorems 4.2.1 and 4.3.1. Moreover, we provide the first known example of a manifold which satisfies the $\partial\overline{\partial}$-lemma but admits non vanishing Aeppli-Bott-Chern-Massey triple products, showing that those products, unlike classical Massey products and Dolbeault products, are not an obstruction to the $\partial\overline{\partial}$-lemma (see Theorem 4.4.1). The construction of the example is performed by taking the quotient of the Iwasawa manifold with respect to a holomorphic action with fixed points, obtaining an orbifold satisfying the $\partial\overline{\partial}$-lemma and then by performing in sequence blow-ups at each singular point so that the resulting object is a smooth complex manifold which still satisfies the $\partial\overline{\partial}$-lemma. We note that such a manifold is not Kähler.

In the fifth chapter, we study the geometrically formal metrics according to Kotschick, geometrically-Dolbeault-formal metrics, and geometrically-Bott-Chern-formal metrics on compact complex surfaces of the class VI and VII of the Enriques-Kodaira classification, namely, the Inoue-Bombieri surfaces and Inoue surfaces of type $\mathcal{S}^{\pm}$, the Hopf surfaces, and the Primary Kodaira and Secondary Kodaira surfaces. In particular, we focus on the study of the stability of the above metric notions under the action of the *Chern-Ricci flow*. Such parabolic geometric flow evolves Hermitian metrics and has been introduced and studied by Gill in [63] as a natural adaptation in the complex setting of the celebrated Ricci flow, which has been the central tool used by Perelman to prove the Poincaré conjecture and the Thurston 's geometrization conjecture in the early 00's. Moreover, the behaviour of the geometric flows at the limits of its time existence usually provides interesting features on the complex structure and topology of the manifold, see [138]. In this setting, we were able to prove that a stability result holds, i.e., the Chern-Ricci flow preserves geometric formalities on any of the above surfaces, see Theorem 5.3.1 and Proposition 5.4.1. This was accomplished by finding an explicit solution of the flow starting from an invariant metric and and then computing the harmonic representatives of de Rham, Dolbeault, and Bott-Chern cohomologies with respect to the solution metric.

In the sixth and final chapter of this thesis work, we focus on the stability under blow-ups of astheno-Kähler metrics and the relation between strong Kähler with torsion metrics and geometrically-Bott-Chern-formal metrics. Voisin has shown that, by blowing up a compact Kähler manifolds at a point or along a complex submanifold, one obtains a Kähler manifold, see [154]. Analogously, the blow-up of either a compact balanced manifold or a compact SKT manifold at a point or along a complex submanifold yields again a balanced, respectively, SKT manifold, see [7] and [55]. Moreover, Fino and Tomassini showed that if the fundamental form ω of a Hermitian metric satisfies certain differential condition, namely

$$\partial\overline{\partial}\omega = 0, \quad \partial\overline{\partial}\omega^2 = 0, \tag{0.0.1}$$

then ω also satisfies $\partial\overline{\partial}\omega^k = 0$, for every k, and properties (0.0.1) are stable under blow-ups at a point or along complex submanifolds. On the other hand, we prove in Theorem 6.2.1 that if the fundamental form associated to a Hermitian metric satisfies weaker conditions than (0.0.1), those condition are not stable under blow-up. In order to do so, we start by explicitly constructing a family of nilmanifolds of complex dimension 5 and characterize the complex structures whose canonical diagonal metric satisfies the astheno-Kähler condition and another suitable differential property, see Theorem 6.2.1. We proceed by selecting a specific element of such a family and a suitable complex submanifold along which we perform a blow-up. We then prove the thesis by using obstructions in [3]. Furthermore, we analize SKT metrics on the Fino-Parton-Salamon manifolds and we show that any invariant metric is also geometrically-Bott-Chern-formal. The same result also applies to any nilmanifold of complex dimension n endowed with a left-invariant complex structure admitting analogous structure equations, see Theorem 6.4.4. We end this chapter, by showing that in general there exists SKT manifolds which do not any geometrically-Bott-Chern-formal metric. The counterexamples provided are products of either two copies of a Inoue-Bombieri surface, two copies of a Primary Kodaira surface, or a copy of a Inoue surface and a Primary Kodaira surface.

The three final appendices are devoted to the definitions of complex manifolds through the a holomorphic atlas, of holomorphic and complex vector bundles (Appendix A), of formality according to Sullivan, of triple Massey products, and the geometrically formal metrics for differentiable manifolds (Appendix B), and the invariant cohomology of real nilmanifolds (Appendix C).

The content of Sections 2.2, 2.4 of Chapter 2, and of Chapters 3 – 5 relies, in order, on the published papers:

- R. Piovani, T. Sferruzza, Deformations of Strong Kähler with torsion metrics, *Complex Manifolds* **8** (2021), 286–301,
- T. Sferruzza, Deformations of balanced metrics, *Bull. Sci. Math.* **178** (2022), 103143,
- T. Sferruzza, N. Tardini, p-Kähler and balanced structures on nilmanifolds with nilpotent complex structures, *Ann. Glob. Anal. Geom.* **62** (2022), 869–881,

- T. Sferruzza, A. Tomassini, Dolbeault and Bott-Chern formalities: deformations and $\partial\overline{\partial}$-lemma, *J. Geom. Phys.* **175** (2022), 104470,
- D. Angella, T. Sferruzza, Geometric formalities along the Chern-Ricci flow, *Complex Anal. Oper. Theory* **14** (2020). $\mathrm{https://doi.org/10.1007/s11785-019-00971-6}$.

The content of Section 2.3 is original work (not submitted) from

- T. Sferruzza, "Deformations of astheno-Kähler metrics",

and the content of Chapter 6 is from the submitted paper

- T. Sferruzza, A. Tomassini, On cohomological and formal properties of strong Kähler with torsion and astheno-Kähler metrics, preprint available at $\mathrm{https://doi.org/10.48550/arXiv.2206.06904}$.

The author would to thank the referees for their comments and suggestions, which helped improve the presentation of this work.

Chapter 1

Preliminaries on complex manifolds and Hermitian structures

In this first chapter, we recall the main facts regarding the cohomological structures, the metric structures, and formal structures of a complex manifold. Unless otherwise stated, we will assume every manifold to be compact. We will approach the topic of complex manifolds from the point of view of even dimensional differentiable manifolds endowed with a integrable almost complex structure; for the definitions of holomorphic atlas, holomorphic maps, and holomorphic bundles, see Appendix A.

1.1 Complex and Hermitian geometry on vector spaces

We start by recalling one of the fundamental structures in complex geometry, namely, the almost complex structure endomorphism on a even dimensional differentiable manifold. Throughout this work, the explicit computations will usually be carried out on manifolds endowed with a structure of compact quotients of real Lie groups with invariant complex structures; such structures will enable to reduce many differential problems to finite dimensional linear algebra problems, therefore, it seemed reasonable to devote this first section to the theory of almost complex structures on vector spaces and then extend the notions here recalled in the later sections.

Let V be a real vector space of dimension $2n$.

Definition 1.1.1. An *almost complex structure* on V is an endormorphism J of V such that $J^2 = -\operatorname{id}_V$.

Clearly, the endomorphism J is invertible, i.e., $J \in \mathrm{GL}(V)$, and the assumption $\dim_{\mathbb{R}} V = 2n$ for some $n \in \mathbb{N}$ is necessary, since if V is a k-dimensional real vector space endowed with an almost complex structure J and $\mathcal{B}$ is a basis for V, then

$$0 < \det(M_{\mathcal{B}}(J))^2 = \det(M_{\mathcal{B}}(J^2)) = \det(-\operatorname{id}_V) = (-1)^k,$$

forcing k to be even.

Any complex vector space admits a natural almost complex structure induced by the multiplication by i. Viceversa, on a $2n$-dimensional real vector space V, an almost complex structure J induces a structure of complex vector space on V in the following way. For every $v \in V$, $a + ib \in \mathbb{C}$, it is sufficient to set

$$(a + ib) \cdot v := av + bJ(v).$$

It is immediate to check that $i^2 \cdot v = J(Jv) = -v$. Thus, the space (V, J) can be regarded as a complex vector space of complex dimension $\dim_{\mathbb{C}}(V, J) = n$. Moreover, every $2n$-dimensional real vector space endowed with an almost complex structure admits an orientation induced by the orientation on $\mathbb{C}^n$.

Example 1.1.2. Let $V = \mathbb{R}^{2n}$ and let $\mathcal{B} = \{e_1, \ldots, e_{2n}\}$ the canonical basis. Then, the position

$$Je_k := \begin{cases} e_{k+n}, & k \in \{1, \ldots, n\}, \\ -e_{k-n}, & k \in \{n+1, \ldots, 2n\}, \end{cases}$$

defines an almost complex structure J on $\mathbb{R}^{2n}$. In particular, as a complex vector space $(\mathbb{R}^{2n}, J)$ coincides with $(\mathbb{C}^n, i)$. The positive orientation on $(\mathbb{R}^{2n}, J)$ is given by the real basis

$$\{e_1, Je_1, \ldots, e_n, Je_n\}.$$

Let now $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the real vector space V. Such a vector space admits a natural complex structure by defining the multiplication

$$z_1 \cdot (v \otimes z_2) := v \otimes (z_1 z_2),$$

for every $z_1 \in \mathbb{C}$, $v \otimes z_2 \in V_{\mathbb{C}}$. Note that V can be viewed as $V \otimes_{\mathbb{R}} 1 \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C}$ and it is clearly a real subspace of $V \otimes_{\mathbb{R}} \mathbb{C}$, i.e., it is invariant under the complex conjugation defined by $\overline{(v \otimes z)} := v \otimes \overline{z}$, for $v \otimes z \in V_{\mathbb{C}}$. A practical way of considering $V_{\mathbb{C}}$ is to identify it with the space $V \oplus iV$, i.e.,

$$V_{\mathbb{C}} := \{v + iw \mid v, w \in V\}.$$

Let us denote by the same symbol the $\mathbb{C}$-linear extension to $V_{\mathbb{C}}$ of the almost complex structure J defined on a $2n$-dimensional real vector space V. Such a endomorphism has complex eigenvalues $\pm i$ on $V_{\mathbb{C}}$, to which correspond the eigenspaces

$$\begin{aligned} V^{1,0} &= \{v \in V_{\mathbb{C}} \,|\, Jv = iv\} \subset V_{\mathbb{C}} \\ V^{0,1} &= \{v \in V_{\mathbb{C}} \,|\, Jv = -iv\} \subset V_{\mathbb{C}}. \end{aligned}$$

Notice that both $V^{1,0}$ and $V^{0,1}$ are complex subspaces of $V_{\mathbb{C}}$ and we have immedaitely the following decomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}. \tag{1.1.1}$$

We remark that if $\dim_{\mathbb{R}} V = 2n$, we have that $\dim_{\mathbb{C}} V_{\mathbb{C}} = 2n$, whereas $\dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1} = n$. By choosing a $\mathbb{C}$-basis $\{v_1, \ldots, v_n\}$ for (V, J), one obtains bases $\mathcal{B}^{1,0}$ and $\mathcal{B}^{0,1}$ for, respectively, $V^{1,0}$ and $V^{0,1}$, by setting

$$\mathcal{B}_{1,0} := \left\{ \frac{1}{2}(v_j - iJv_j) \right\}_{j=1,\ldots,n} \tag{1.1.2}$$

$$\mathcal{B}_{0,1} := \left\{ \frac{1}{2}(v_j + iJv_j) \right\}_{j=1,\ldots,n}. \tag{1.1.3}$$

We remark that, since $\dim_{\mathbb{C}}(V, J) = \dim_{\mathbb{C}} V^{1,0} = n$ and the $\mathbb{C}$-linear map

$$\begin{aligned} (V, J) &\to V^{1,0} \\ v &\mapsto \frac{1}{2}(v - iJv) \end{aligned} \tag{1.1.4}$$

is bijective, we have that (V, J) and $V^{1,0}$ are isomorphic as complex vector spaces.

If V^* denotes the dual vector space of a vector space V, an almost complex structure on V naturally induces an almost complex structure on V^*, which will be still denoted by J. In fact, if $\eta \in V^*$, $v \in V$, it suffices to set

$$J\eta\,(v) := \eta(Jv).$$

By considering the complexification of V^*, i.e., $V^*_{\mathbb{C}} := V^* \otimes \mathbb{C}$, and the $\mathbb{C}$-linear extension of J to such space, one can define the eigenspaces with respect to the $\pm i$ eigenvalues

$$(V^*)^{1,0} = \{\eta \in V^*_{\mathbb{C}} \mid J\eta = i\eta\} \subset V^*_{\mathbb{C}}$$
$$(V^*)^{0,1} = \{\eta \in V^*_{\mathbb{C}} \mid J\eta = -i\eta\} \subset V^*_{\mathbb{C}},$$

and a decomposition analogous to (1.1.1) holds

$$V^*_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}.$$

As complex vector spaces, $(V^*)^{1,0}$ and $(V^*)^{0,1}$ admit bases as in (1.1.2) and (1.1.3). However, it is sometimes useful to work with dual bases, in the following way. Let $\{v^1, \dots, v^n\}$ be the $\mathbb{C}$-basis of (V^*, J) dual to the $\mathbb{C}$-basis $\{v_1, \dots, v_n\}$ of (V, J), i.e., such that

$$v^i(v_j) = \delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

The dual bases $\mathcal{B}^{1,0}$ and $\mathcal{B}^{0,1}$ to, respectively, $\mathcal{B}_{1,0}$ and $\mathcal{B}_{0,1}$, are then given by

$$\mathcal{B}^{1,0} := \left\{v^i + iJv^i\right\}_{i=1,\dots,n}$$
$$\mathcal{B}^{0,1} := \left\{v^i - iJv^i\right\}_{i=1,\dots,n},$$

Let us now consider the *k-covectors* on V, i.e., the elements of the exterior powers $\bigwedge^k V^*_{\mathbb{C}}$ of the complexified dual vector space $V^*_{\mathbb{C}}$ of a real vector space V. If V is endowed with an almost complex structure J, the spaces $\bigwedge^k V^*_{\mathbb{C}}$ admit the following decompositions in terms of the $\pm i$-eigenspaces $(V^*)^{1,0}$ and $(V^*)^{0,1}$ of J on $V^*_{\mathbb{C}}$, namely,

$$\textstyle\bigwedge^k V^*_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V, \tag{1.1.5}$$

where we set

$$\textstyle\bigwedge^{p,q} V := \bigwedge^p (V^*)^{1,0} \otimes \bigwedge^q (V^*)^{0,1},$$

for the space of *(p,q)-covectors*. In particular, if $\{\eta^1, \dots, \eta^n\}$ is a basis of $(V^{1,0})^*$, it is easy to see that the set

$$\{\eta^{i_1} \wedge \dots \wedge \eta^{i_p} \wedge \overline{\eta^{j_1}} \wedge \dots \wedge \overline{\eta^{j_q}} \mid 1 \leq i_1 < \dots < i_p \leq n,\ 1 \leq j_1 < \dots < j_q \leq n\}$$

is a basis of $\bigwedge^{p,q} V$. Note that one can extend the action of the almost complex structure J to the space of k-covectors in $\bigwedge^k(V^*_{\mathbb{C}})$ by setting, for any given $\alpha \in \bigwedge^k(V^*_{\mathbb{C}})$,

$$J\alpha(v_1, \dots, v_k) := \alpha(Jv_1, \dots, Jv_k). \tag{1.1.6}$$

However, such a J is not necessarily an almost complex structure on $\bigwedge^k(V^*_{\mathbb{C}})$.

From the metric point of view, one may want to define scalar products which are compatible with the almost complex structure endomorphism on a vector space.

More precisely, let V be a $2n$-dimensional real vector space endowed a positive definite scalar product g.

Definition 1.1.3. An almost complex structure J on V is said to be *compatible* with g if for every $v, w \in V$, it holds

$$g(Jv, Jw) = g(v, w),$$

i.e., J is an isometry of V with respect to g.

Given a scalar product g on V, it is always possible to extend it to either a Hermitian scalar product g_H, or to a $\mathbb{C}$-bilinear scalar product $g_{\mathbb{C}}$ on $V_{\mathbb{C}}$ as follows

$$g_H(v_1 + iv_2, w_1 + iw_2) := g(v_1, w_1) + g(v_2, w_2) + ig(v_2, w_1) - ig(v_1, w_2), \tag{1.1.7}$$

$$g_{\mathbb{C}}(v_1 + iv_2, w_1 + iw_2) := g(v_1, w_1) + g(v_2, w_2) - ig(v_2, w_1) - ig(v_1, w_2), \tag{1.1.8}$$

for every $v_1 + iv_2, w_1 + iw_2 \in V_{\mathbb{C}}$. Note that if V is endowed with an almost complex structure J compatible with g, the decomposition induced by J on $V_{\mathbb{C}}$ given by (1.1.5) is orthogonal with respect to g_H.

Let V be real vector space endowed with a positive definite scalar product g and a compatible almost complex structure J.

Definition 1.1.4. The *fundamental form* ω associated to g is the positive 2-covector defined by

$$\omega(v, w) := g(Jv, w),$$

for every $v, w, \in V$.

As by definition, ω is alternating, i.e., $\omega \in \bigwedge^2 V^*$ and, if such form is naturally extended to $\bigwedge^2 V^*_{\mathbb{C}}$ by

$$\omega(v_1 + iv_2, w_1 + iw_2) := g_{\mathbb{C}}(J(v_1 + iv_2), w_1 + iw_2)$$

is it clear that $\omega \in \bigwedge^{1,1} V$ and $\overline{\omega} = \omega$, i.e., the fundamental form ω of a positive definite scalar product g is a *real* $(1,1)$*-covector* on V. Moreover, since $\omega(v, Jv) = g(v, v) \geq 0$, with equality holding if, and only if, $v = 0$, the covector ω is also *positive*.

Remark 1.1.5. If g is positive definite scalar product on V and J is compatible with g, then it can be easily seen that the form

$$h := g - i\omega \tag{1.1.9}$$

is a positive Hermitian scalar product on (V, J). Viceversa, if h is a positive definite Hermitian scalar product on (V, J), then

$$\mathfrak{Re}(h) \colon V \times V \to \mathbb{R}$$

is a positive definite scalar product on V and

$$-\mathfrak{Im}(h) \colon V \times V \to \mathbb{R}$$

is a positive 2-covector on V.

If V is a real vector space endowed with an almost complex structure J and a compatible positive definite scalar product g, then via the isomorphism (1.1.4), the Hermitian extension g_H and h as defined in (1.1.9) satisfy the following relation

$$g_H|_{V^{1,0}} = \frac{1}{2}h. \tag{1.1.10}$$

Equation (1.1.10) is useful for computing the coordinate expression of ω in terms of the coordinate expression of $g_H|_{V^{1,0}}$. Let $\{z_j := \frac{1}{2}(x_j - iJx_j)\}_{j=1}^n$ be a $\mathbb{C}$-basis for $V^{1,0}$. If we set $y_j := Jx_j$, then

$\{x_1, y_1, \ldots, x_n, y_n\}$ is a $\mathbb{R}$-basis for V, whereas $\{x_1, \ldots, x_n\}$ is a $\mathbb{C}$-basis for (V, J), via (1.1.4). Let us now assume that the product $g_H|_{V^{1,0}}$ is expressed locally by

$$g_H|_{V^{1,0}} = \frac{1}{2} \sum_{j,k=1}^{n} g_{j\bar{k}} z^j \otimes z^k,$$

where $\{z^j\}_{j=1}^n$ is the dual basis of $\{z_j\}_{j=1}^n$ and $g_{j\bar{k}} \in \mathbb{C}$. Since g_H is Hermitian, we have that $g_{j\bar{j}} \in \mathbb{R}$ and $g_{j\bar{k}} = \overline{g_{k\bar{j}}}$.

The fundamental form ω of g is a 2-form on V, therefore it can be written as

$$\omega = \sum_{j,k=1}^{n} \omega(x_j, y_k) x^j \wedge y^k + \sum_{j<k} \omega(x_j, x_k) x^j \wedge x^k + \sum_{j<k} \omega(y_j, y_k) y^j \wedge y^k.$$

From $\omega = -\mathfrak{Im}(h)$ and (1.1.10), we have that $\omega(x_j, x_k) = \omega(y_j, y_k) = -\mathfrak{Im}(g_{i\bar{j}})$, and $\omega(x_j, y_k) = \mathfrak{Re}(g_{i\bar{j}})$. Hence

$$\omega = \sum_{j,k=1}^{n} \mathfrak{Re}(g_{j\bar{k}}) x^j \wedge y^k - \sum_{j<k} \mathfrak{Im}(g_{j\bar{k}})(x^j \wedge x^k + y^j \wedge y^k).$$

Since $\mathfrak{Re}(g_{j\bar{j}}) = g_{j\bar{j}}$ and $\mathfrak{Re}(g_{j\bar{k}}) = \mathfrak{Re}(g_{k\bar{j}})$, one has that

$$\omega = \sum_{j=1}^{n} g_{j\bar{j}} x^j \wedge y^j + \sum_{j<k} \mathfrak{Re}(g_{j\bar{k}})(x^j \wedge y^k + x^k \wedge y^j) - \sum_{j<k} \mathfrak{Im}(g_{j\bar{k}})(x^j \wedge x^k + y^j \wedge y^k).$$

If one consider the $\mathbb{C}$-linear extension of ω to $\bigwedge^{1,1} V$ and exploits the relations

$$\begin{aligned} x^j \wedge x^k + y^j \wedge y^k &= \frac{1}{2}(z^j \wedge \bar{z}^k + \bar{z}^j \wedge z^k), \\ x^j \wedge y^j &= \frac{i}{2}(z^j \wedge \bar{z}^j), \\ x^j \wedge y^k + x^k \wedge y^j &= \frac{i}{2}(z^j \wedge \bar{z}^k - \bar{z}^j \wedge z^k), \end{aligned}$$

it turns out that

$$\begin{aligned} \omega &= \frac{i}{2} \sum_{j=1}^{n} g_{j\bar{j}} z^j \wedge \bar{z}^j + \frac{i}{2} \sum_{j<k} \mathfrak{Re}(g_{j\bar{k}})(z^j \wedge \bar{z}^k - \bar{z}^j \wedge z^k) - \frac{1}{2} \sum_{i<j} \mathfrak{Im}(g_{j\bar{k}})(z^j \wedge \bar{z}^k + \bar{z}^j \wedge z^k) \\ &= \frac{i}{2} \sum_{j=1}^{n} g_{j\bar{j}} z^j \wedge \bar{z}^j + \frac{i}{2} \sum_{j<k} (\mathfrak{Re}(g_{j\bar{k}}) + i\mathfrak{Im}(g_{j\bar{k}})) z^j \wedge \bar{z}^k + \frac{i}{2} \sum_{j<k} (\mathfrak{Re}(g_{j\bar{k}}) - i\mathfrak{Im}(g_{j\bar{k}})) z^k \wedge \bar{z}^j \\ &= \frac{i}{2} \sum_{j=1}^{n} g_{j\bar{j}} z^j \wedge \bar{z}^j + \frac{i}{2} \sum_{j<k} g_{j\bar{k}} z^j \wedge \bar{z}^k + \frac{i}{2} \sum_{j<k} \overline{g_{j\bar{k}}} z^k \wedge \bar{z}^j \\ &= \frac{i}{2} \sum_{j=1}^{n} g_{j\bar{j}} z^j \wedge \bar{z}^j + \frac{i}{2} \sum_{j<k} g_{j\bar{k}} z^j \wedge \bar{z}^k + \frac{i}{2} \sum_{j<k} g_{k\bar{j}} z^k \wedge \bar{z}^j \\ &= \frac{i}{2} \sum_{i,j=1}^{n} g_{j\bar{k}} z^j \wedge \bar{z}^k. \end{aligned}$$

Then, if ω is expressed as $\omega = \frac{1}{2} \sum_{j,k=1}^{n} \omega_{j\bar{k}} z^j \wedge \bar{z}^j$, the relation between the coeffiecients of ω and $g_H|_{V^{1,0}}$ is given by

$$\omega_{j\bar{k}} = i \cdot g_{j\bar{k}}.$$

In the last part of this section, we recall the definition of the Hodge $*$-operator for complex vector spaces, both in the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear version.

We start by remarking that a scalar product g on a vector space V induces a scalar product on V^* in the following way. Let $\mathcal{B}$ be a basis for V and let $\{g_{ij}\}$ be the matrix representing g with respect to $\mathcal{B}$. If $\mathcal{B}^*$ is the basis of V^* dual to $\mathcal{B}$, then the matrix

$$\{g^{ij}\} := \{g_{ij}\}^{-1}$$

represents a scalar product g on V^* with respect to the basis $\mathcal{B}^*$.

Then, one can consider the extension of g to the Hermitian scalar product g_H on $V^*_{\mathbb{C}}$ as in (1.1.7) and then extend g_H to a Hermitian product on $\bigwedge^k V^*_{\mathbb{C}}$, in the following way. Let $\{v^1, \dots, v^{2n}\}$ be a $\mathbb{C}$-basis for $V^*_{\mathbb{C}}$. For $I = \{1 \le i_1 < \dots < i_k \le 2n\}$, $J = \{1 \le j_1 < \dots < j_k \le 2n\}$, we set $v^I := v^{i_1} \wedge \dots \wedge v^{i_k}$ and v^J analogously. The position

$$g_H(v^I, v^J) := \det\left(\{g_H(v^{i_l}, v^{j_m})\}_{l,m=1}^k\right)$$

then yields a Hermitian scalar product on $\bigwedge^k V^*_{\mathbb{C}}$.

If the vector space V^* is endowed with an almost complex structure J and, hence, there holds a decomposition of type (1.1.5) in (p,q)-covectors, in a similar way the scalar product g can be extended to a Hermitian scalar product g_H on each space $\bigwedge^{p,q} V$.

Let V be a real vector space of dimension $2n$ endowed with a positive definite scalar product g and let J be an almost complex structure compatible with g and ω the associated fundamental form of g.

Definition 1.1.6. The $\mathbb{C}$-*linear Hodge* $*$-*operator* is defined at the level of (p,q)-forms on (V, J) by

$$\begin{aligned} *\colon \textstyle\bigwedge^{p,q} V &\to \textstyle\bigwedge^{n-q,n-p} V \\ \beta &\mapsto *\beta, \end{aligned}$$

where $\alpha \wedge \overline{*\beta} := g_H(\alpha, \beta)\,\mathrm{vol}$, for every $\alpha \in \bigwedge^{p,q} V$ and $\mathrm{vol} = \frac{\omega^n}{n!}$ is the volume covector, i.e., a positive $2n$-covector on V, naturally induced by g.

As the usual Hodge $*$-operator, $*$ is an an isometry with respect to g_H, it is self-adjoint up to a sign and involutive up to a sign. Analogously, one defines the $\mathbb{C}$-*antilinear Hodge* $*$-*operator* as

$$\begin{aligned} *\colon \textstyle\bigwedge^{p,q} V &\to \textstyle\bigwedge^{n-p,n-q} V \\ \beta &\mapsto *\beta \end{aligned}$$

where $\alpha \wedge *\beta := g_H(\alpha, \beta)\,\mathrm{vol}$.

1.2 Complex and Hermitian structures on manifolds

One of the most important objects associated to differentiable manifold M is the tangent bundle TM. At each point $p \in M$, the tangent space T_pM is a real dimensional vector space such that $\dim_{\mathbb{R}} T_pM = \dim_{\mathbb{R}} M$. Definitions from section 1.1 can then be extended to the tangent bundle of even-dimensional differentiable manifolds by requiring that the objects vary pointwisely smoothly on the manifold.

Let M be a differentiable manifold of real dimension $2n$ and let TM be its tangent bundle.

Definition 1.2.1. An *almost complex structure* on M is an endomorphism $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{id}_{TM}$. Such an almost complex structure J on M is said to be *integrable* if it is induced by holomorphic coordinates, see Appendix A.

By the Newlander-Niremberg theorem [107], the integrability of an almost complex structure J is equivalent to the vanishing of the *Nijenhuis tensor* N_J associated to J, that is, J is integrable if, and only, if

$$N_J(X,Y) := [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y] = 0, \tag{1.2.1}$$

for every $X, Y \in TM$. Other equivalent conditions of integrability for an almost complex structure will be recalled later in this section.

We will denote the complex manifold arising from assigning the integrable almost complex structure J on a $2n$-dimensional differentiable manifold M by (M,J), unless the almost complex structure has already been fixed; in that case, we will denote it simply by M. Note that $\dim_{\mathbb{C}}(M,J) = n$.

One can extend the endomorphism J by $\mathbb{C}$-linearity to the complexified tangent bundle $T_{\mathbb{C}}M := TM \otimes \mathbb{C}$ and obtain a decomposition in terms of the $\pm i$-eigenspace bundles of J, namely

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M = \{Z \in T_{\mathbb{C}}M \mid (J - iI)Z = 0\}$ and $T^{0,1}M = \{Z \in T_{\mathbb{C}}M \mid (J + iI)Z = 0\}$. Also, if J is extended to the complexified cotangent bundle $T^*_{\mathbb{C}}M$, one similarly obtains

$$T^*_{\mathbb{C}}M = (T^{1,0}M)^* \oplus (T^{0,1}M)^* \tag{1.2.2}$$

where $(T^{1,0}M)^*$ and $(T^{0,1}M)^*$ are the eigenspaces bundles associated, respectively, to the eigenvalues i and $-i$.

Furthermore, on the exterior powers $\wedge^k_{\mathbb{C}} M := \wedge^k(T^*_{\mathbb{C}}M)$, the decomposition (1.2.2) induces the following

$$\wedge^k_{\mathbb{C}} M = \bigoplus_{p+q=k} \wedge^{p,q} M, \tag{1.2.3}$$

where $\wedge^{p,q} M := \wedge^p(T^{1,0}M)^* \otimes \wedge^q(T^{0,1}M)^*$. We will denote the spaces of the global sections of the bundles $\wedge^k_{\mathbb{C}} M$ and $\wedge^{p,q} M$, i.e., the spaces of k-*complex forms* on M (or *forms of degree* k) and of (p,q)-*forms* on M (or *forms of bedegree* (p,q)), by respectively, $\mathcal{A}^k_{\mathbb{C}}(M)$ and $\mathcal{A}^{p,q}(M)$.

The action of the exterior differential d can be extended to complex forms and maps k-complex forms into $(k+1)$-complex forms, namely

$$d\colon \mathcal{A}^k_{\mathbb{C}}(M) \to \mathcal{A}^{k+1}_{\mathbb{C}}(M).$$

With respect to any generic almost complex structure J on M, at the level of (p,q)-forms, the exterior differential d acts as

$$d\colon \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p+2,q-1}(M) \oplus \mathcal{A}^{p+1,q}(M) \oplus \mathcal{A}^{p,q+1}(M) \oplus \mathcal{A}^{p-1,q+2}(M),$$

i.e., by denoting the projections $\mu := \pi^{p+2,q-1} \circ d$, $\partial := \pi^{p+1,q} \circ d$, $\overline{\partial} := \pi^{p,q+1} \circ d$, and $\overline{\mu} := \pi^{p-1,q+2} \circ d$, it holds that d splits as

$$d = \mu + \partial + \overline{\partial} + \overline{\mu}.$$

Since $d^2 = 0$ and decomposition (1.2.3) is direct, it follows immediately that

$$\begin{aligned} \mu^2 &= 0 \\ \mu\partial + \partial\mu &= 0 \\ \partial^2 + \mu\overline{\partial} + \overline{\partial}\mu &= 0 \\ \mu\overline{\mu} + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\mu}\mu &= 0 \\ \overline{\partial}^2 + \overline{\mu}\partial + \partial\overline{\mu} &= 0 \end{aligned}$$

$$\begin{aligned}\overline{\mu}\overline{\partial} + \overline{\partial}\overline{\mu} &= 0\\ \overline{\mu}^2 &= 0.\end{aligned}$$

It is fairly easy to see that, in addition to the vanishing of the Nijenhuis tensor associated to J, necessary and sufficient conditions under which the almost complex structure J is integrable are either of the following

1. $\mu = \overline{\mu} = 0$,

2. $\overline{\mu}|_{\mathcal{A}^{1,0}(M)} = 0$, that is $\pi^{0,2}(d\alpha) = 0$, for any $\alpha \in \mathcal{A}^{1,0}(M)$,

3. $[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$.

In any of the previous situations, it holds that

$$d\colon \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p+1,q} \oplus \mathcal{A}^{p,q+1}(M),$$

i.e., d decomposes as

$$d = \partial + \overline{\partial},$$

with $\partial\overline{\partial} = -\overline{\partial}\partial$ and $\partial^2 = \overline{\partial}^2 = 0$. From now on, unless specified, we will assume J to be integrable.

It is sometimes useful to define the following operator

$$d^c := J^{-1} \circ d \circ J, \tag{1.2.4}$$

or equivalently, $d^c = -i(\partial - \overline{\partial})$, for which it is immediate to see that

$$dd^c = -d^c d = 2i\partial\overline{\partial}. \tag{1.2.5}$$

Let now g be a Riemannian metric on M and let J be an integrable almost complex structure on M.

Definition 1.2.2. The metric g is said a *Hermitian metric* on (M, J) if g is compatible with J, i.e., $g(JX, JY) = g(X, Y)$, for every $X, Y \in TM$. The *fundamental form* ω of g is the 2-form defined by

$$\omega(X, Y) = g(JX, Y),$$

for every $X, Y \in TM$.

In particular, if ω is extended by $\mathbb{C}$-linearity to $\mathcal{A}^2_{\mathbb{C}}(M)$, then ω is a form of bidegree $(1,1)$ and is real, i.e., $\omega \in \mathcal{A}^{1,1}(M)$ and $\omega = \overline{\omega}$.

Remark 1.2.3. If g is Hermitian metric on (M, J) and ω is its fundamental form, then any two structures of $\{J, g, \omega\}$ determine the remaining one.

A complex manifold (M, J) endowed with a Hermitian metric g with fundamental associated ω will be referred to a *Hermitian manifold*, and it will be denoted as (M, J, g, ω). Note that the form $\frac{\omega^n}{n!}$ naturally determines a *volume* on M, i.e., a everywhere non vanishing (n, n)-form, or equivalently, an orientation on (M, J). Therefore, every Hermitian manifold is orientable.

Let now $\pi\colon E \to M$ be a holomorphic vector bundle of rank r over a complex manifold (M, J) (see Appendix A).

Definition 1.2.4. The bundle of *k-complex forms on M with values in E* is the bundle

$$\wedge^k_{\mathbb{C}}(M,E) := \wedge^k_{\mathbb{C}} M \otimes E,$$

and the bundle of *(p,q)-forms with values in E* is the bundle

$$\wedge^{p,q}(M,E) = \wedge^{p,q} M \otimes E.$$

The global section of such bundles will be denoted by, respectively, $\mathcal{A}^k_{\mathbb{C}}(M)$ and $\mathcal{A}^{p,q}(M,E)$; the symbol "M" will be omitted whenerever the situation is clear.

A special case of such bundles is given by $E = T^{1,0}M$. In this situation, an element $\psi \in \mathcal{A}^{0,q}(T^{1,0})$ will be called a *$(0,q)$-vector form on M*.

Let now $\pi\colon E \to M$ be a holomorphic vector bundle on M. Then, one can define the interior product between a $(0,1)$-vector form on M and any (r,s)-form with values in E. Let $\psi \in \mathcal{A}^{0,1}(T^{1,0}M)$, that is $\psi = \eta \otimes Z$, with $\eta \in \mathcal{A}^{0,1}(M)$, $Z \in T^{1,0}M$, and $\beta \otimes s \in \mathcal{A}^{r,s}(E)$. Then, the *interior product* of ψ and $\beta \otimes s$ is given by

$$\begin{aligned} i\colon \mathcal{A}^{r,s}(E) &\to \mathcal{A}^{r-1,s+1}(E) \\ i_\psi(\beta \otimes s) &:= \eta \wedge \iota_Z(\beta) \otimes s, \end{aligned} \tag{1.2.6}$$

where $i_Z\colon \mathcal{A}^{r,s}(M) \to \mathcal{A}^{r-1,s}(M)$ is the usual contraction of differential forms by the vector field Z. The position (1.2.6) can then be extended by linearity to any $\psi \in \mathcal{A}^{p,q}(T^{1,0}M)$ and $\alpha \in \mathcal{A}^{r,s}(E)$. Also, it is possible to define $i_{\overline{\psi}}(\beta \otimes s) = \overline{\eta} \wedge i_{\overline{Z}}(\alpha) \otimes s$ for the conjugate $\overline{\psi} = \overline{\eta} \otimes \overline{Z} \in \mathcal{A}^{1,0}(T^{0,1}M)$. We will also denote the map i_φ by the symbol $\varphi \lrcorner$.

1.3 Complex cohomologies and Hodge Theory

Let (M,J) be a compact complex n-dimensional manifold. Among the main invariants associated to the complex structure J of (M,J), there are the following cohomology spaces ([32, 1])

$$\text{Dolbeault cohomology} := H^{p,q}_{\overline{\partial}}(M) := \frac{\operatorname{Ker}(\overline{\partial}\colon \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M))}{\operatorname{Im}(\overline{\partial}\colon \mathcal{A}^{p,q-1}(M) \to \mathcal{A}^{p,q}(M))} \tag{1.3.1}$$

$$\text{Bott-Chern cohomology} := H^{p,q}_{BC}(M) := \frac{\operatorname{Ker}(d\colon \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}(M) \oplus \mathcal{A}^{p,q+1}(M))}{\operatorname{Im}(\partial\overline{\partial}\colon \mathcal{A}^{p-1,q-1}(M))} \tag{1.3.2}$$

$$\text{Aeppli cohomology} := H^{p,q}_{A}(M) := \frac{\operatorname{Ker}(\partial\overline{\partial}\colon \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p+1,q+1}(M))}{\operatorname{Im}(\partial\colon \mathcal{A}^{p-1,q}(M)) + \operatorname{Im}(\overline{\partial}\colon \mathcal{A}^{p,q-1}(M))}. \tag{1.3.3}$$

We point out that, since M is compact, all the spaces above are finite dimensional; we set $h^{p,q}_\sharp(M) = \dim H^{p,q}_\sharp(M)$, for $\sharp \in \{\overline{\partial}, BC, A\}$.

The cohomologies spaces (1.3.1), (1.3.2), and (1.3.3) arise as cohomologies of certain complex of differential forms on the manifold (M,J). In particular, the Dolbeault cohomology is the "column" cohomology of the complex of (p,q)-forms $\mathcal{A}^{\bullet,\bullet}(M)$ endowed with the $\overline{\partial}$ operator, i.e., for every p,

$$\mathcal{A}^{p,0}(M) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1}(M) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \mathcal{A}^{p,n}(M)$$

and

$$H^{p,\bullet}_{\overline{\partial}}(M) = H^\bullet(\mathcal{A}^{p,\bullet}(M), \overline{\partial}),$$

whereas the Bott-Chern cohomology and the Aeppli-cohomology spaces of a fixed bedegree (p,q) coincide with the cohomology of the complexes, respectively,

$$\mathcal{A}^{p-1,q-1}(M) \xrightarrow{\partial\overline{\partial}} \mathcal{A}^{p,q}(M) \xrightarrow{d} \mathcal{A}^{p+1,q}(M) \oplus \mathcal{A}^{p,q+1}(M)$$

and

$$\mathcal{A}^{p-1,q}(M) \oplus \mathcal{A}^{p,q-1}(M) \xrightarrow{d} \mathcal{A}^{p,q}(M) \xrightarrow{\partial\overline{\partial}} \mathcal{A}^{p+1,q+1}(M).$$

As a direct consequence of the definitions, the following diagram of natural maps between Dolbeault, de Rham, Bott-Chern, and Aeppli cohomology, is well defined

$$\begin{tikzcd} & H^{\bullet,\bullet}_{BC}(M) \arrow[dl] \arrow[d] \arrow[dr] & \\ H^{\bullet,\bullet}_{\overline{\partial}}(M) \arrow[dr] & H^{\bullet+\bullet}_{dR}(M;\mathbb{C}) \arrow[d] & H^{\bullet,\bullet}_{\partial}(M) \arrow[dl] \\ & H^{\bullet,\bullet}_{A}(M) & \end{tikzcd} \tag{1.3.4}$$

where $H^{\bullet,\bullet}_{\partial}(M)$ is the conjugate cohomology of $H^{\bullet,\bullet}_{\overline{\partial}}(M)$.

A priori, the maps are neither injective nor surjective. However, if the maps are all isomorphisms, we say that (M,J) *satisfies the* $\partial\overline{\partial}$*-lemma*. There exist many equivalent conditions under which a complex manifold (M,J) satisfies the $\partial\overline{\partial}$-lemma, e.g., the injectivity of any of the maps of diagram (1.3.4). For example, (M,J) satisfies the $\partial\overline{\partial}$-lemma, if, and only if,

$$\operatorname{Ker} d \cap \operatorname{Im} \overline{\partial} \subset \operatorname{Im} \partial\overline{\partial}.$$

Furthermore, recently, Angella and Tomassini in [19, Theorem B] (see also [17]) provided a numerical necessary and sufficient condition for the validity of the $\partial\overline{\partial}$-lemma, involving the dimensions of the Bott-Chern and Aeppli cohomologies, i.e., for every k it must hold that

$$\sum_{p+q=k} h^{p,q}_{BC} + h^{p,q}_{A} = 2b_k,$$

A class of complex manifolds satisfying the $\partial\overline{\partial}$-lemma is the class of compact Kähler manifolds, where a *Kähler manifold* is an even dimensional differentiable manifold M endowed with a *Kähler structure* $\{g,J\}$, where J is an integrable almost complex structure and g is a Hermitian metric on (M,J) such that its fundamental form ω is d-closed, i.e,

$$d\omega = 0.$$

Remark 1.3.1. We remark the following behaviors of the cohomology spaces under the action of the complex conjugation and the Hodge $*$-operator:

- $\overline{H^{p,q}_{\overline{\partial}}(M)} = H^{q,p}_{\partial}(M)$, but in general $\overline{H^{p,q}_{\overline{\partial}}(M)} \neq H^{q,p}_{\overline{\partial}}(M)$,
- $\overline{H^{p,q}_{BC}(M)} = H^{q,p}_{BC}(M)$ and $\overline{H^{p,q}_{A}(M)} = H^{q,p}_{A}(M)$,
- $*H^{p,q}_{\overline{\partial}}(M) = H^{n-q,n-p}_{\overline{\partial}}(M)$ and $*H^{p,q}_{BC}(M) = H^{n-q,n-p}_{A}(M)$, where $*$ is the $\mathbb{C}$-linear Hodge $*$-operator,
- the spaces $H^{p,q}_{\overline{\partial}}(M)$ and $H^{p,q}_{BC}(M)$ have a structure of algebra induced by the $\cup$ product of cohomology classes, which for any two cohomology classes $[\alpha],[\beta]$, it is defined as $[\alpha]\cup[\beta] := [\alpha\wedge\beta]$,
- the spaces $H^{p,q}_{A}(M)$ have a structure of $H^{p,q}_{BC}(M)$-module induced by the $\cup$ product.

Let us now consider the compact Hermitian manifold (M, J, g, ω). The Hermitian metric g defines a Hermitian product on each $\mathcal{A}^{p,q}(M)$

$$(\alpha, \beta) := \int_M \alpha \wedge \overline{*\beta}. \tag{1.3.5}$$

Via the Hodge $*$-operator, it is possible to define the adjoint operators of ∂ and $\overline{\partial}$ with respect to $(\cdot, \cdot)$ by

$$\partial^* := - * \circ \overline{\partial} \circ *, \qquad \overline{\partial}^* := - * \circ \partial \circ *.$$

Notice that ∂^* and $\overline{\partial}^*$ are operators of type, respectively, $(-1, 0)$ and $(0, -1)$. We recall the expressions of the *Dolbeault Laplacian*, *Bott-Chern Laplacian*, and *Aeppli Laplacian* (see [129])

$$\begin{aligned}
\Delta_{\overline{\partial}} &= \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}, \\
\Delta_{BC} &= \partial\overline{\partial}\overline{\partial}^*\partial^* + \overline{\partial}^*\partial^*\partial\overline{\partial} + \overline{\partial}^*\partial\partial^*\overline{\partial} + \partial^*\overline{\partial}\overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial} + \partial^*\partial, \\
\Delta_A &= \partial\partial^* + \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\partial^*\partial\overline{\partial} + \partial\overline{\partial}\overline{\partial}^*\partial^* + \partial\overline{\partial}^*\overline{\partial}\partial^* + \overline{\partial}\partial^*\partial\overline{\partial}^*,
\end{aligned}$$

which are self-adjoint with respect to $(\cdot, \cdot)$ as in (1.3.5), elliptic differential operators on each space $\mathcal{A}^{p,q}(M)$. The spaces of *Dolbeault harmonic forms*, *Bott-Chern harmonic forms*, and *Aeppli harmonic forms* are defined as the kernel of such operators, i.e.,

$$\mathcal{H}^{p,q}_{\Delta_{\overline{\partial}}}(M) = \{\alpha \in \mathcal{A}^{p,q}(M) : \Delta_{\overline{\partial}}\alpha = 0\} \tag{1.3.6}$$

$$\mathcal{H}^{p,q}_{\Delta_{BC}}(M) = \{\alpha \in \mathcal{A}^{p,q}(M) : \Delta_{BC}\alpha = 0\} \tag{1.3.7}$$

$$\mathcal{H}^{p,q}_{\Delta_A}(M) = \{\alpha \in \mathcal{A}^{p,q}(M) : \Delta_A\alpha = 0\}. \tag{1.3.8}$$

Note that the same symmetries under complex conjugation and the Hodge $*$-operator pointed out in Remark 1.3.1 hold true also for the spaces of Dolbeault-, Bott-Chern, and Aeppli-harmonic forms.

Since M is compact and each Laplacian is elliptic, (1.3.6), (1.3.7), and (1.3.8) are finite dimensional and there holds a Hodge decomposition for every case (see [129]), namely

$$\begin{aligned}
\mathcal{A}^{p,q}(M) &= \operatorname{Im}\overline{\partial}|_{\mathcal{A}^{p,q-1}(M)} \,\oplus\, \mathcal{H}^{p,q}_{\Delta_{\overline{\partial}}}(M) \,\oplus\, \operatorname{Im}\overline{\partial}^*|_{\mathcal{A}^{p,q+1}(M)} \\
\mathcal{A}^{p,q}(M) &= \operatorname{Im}\partial\overline{\partial}|_{\mathcal{A}^{p-1,q-1}(M)} \,\oplus\, \mathcal{H}^{p,q}_{\Delta_{BC}}(M) \,\oplus\, \left(\operatorname{Im}\partial^*|_{\mathcal{A}^{p+1,q}(M)} + \operatorname{Im}\overline{\partial}^*|_{\mathcal{A}^{p,q+1}(M)}\right) \\
\mathcal{A}^{p,q}(M) &= \operatorname{Im}(\partial\overline{\partial})^*|_{\mathcal{A}^{p+1,q+1}(M)} \,\oplus\, \mathcal{H}^{p,q}_{\Delta_A}(M) \,\oplus\, \left(\operatorname{Im}\partial|_{\mathcal{A}^{p-1,q}(M)} + \operatorname{Im}\overline{\partial}|_{\mathcal{A}^{p,q-1}(M)}\right).
\end{aligned}$$

Moreover, the following maps are isomorphisms of vector spaces (not necessarily of algebras or H_{BC}-modules)

$$\mathcal{H}^{p,q}_{\Delta_{\overline{\partial}}}(M) \xrightarrow{\sim} H^{p,q}_{\overline{\partial}}(M), \qquad \mathcal{H}^{p,q}_{\Delta_{BC}}(M) \xrightarrow{\sim} H^{p,q}_{BC}(M), \qquad \mathcal{H}^{p,q}_{\Delta_A}(M) \xrightarrow{\sim} H^{p,q}_{A}(M),$$

and the harmonic forms with respect to each Laplacian can be characterized as follows

$$\alpha \in \mathcal{H}_{\Delta_{\overline{\partial}}} \iff \begin{cases} \overline{\partial}\alpha = 0 \\ \overline{\partial}^*\alpha = 0 \end{cases} \tag{1.3.9}$$

$$\alpha \in \mathcal{H}_{\Delta_{BC}} \iff \begin{cases} \partial\alpha = 0 \\ \overline{\partial}\alpha = 0 \\ \partial\overline{\partial} * \alpha = 0 \end{cases} \tag{1.3.10}$$

$$\alpha \in \mathcal{H}^{p,q}_A(M) \iff \begin{cases} \partial\overline{\partial}\alpha = 0 \\ \partial * \alpha = 0 \\ \overline{\partial} * \alpha = 0 \end{cases} . \tag{1.3.11}$$

Hodge theory, with slight due changes, can be applied to the setting of Dolbeault cohomology with values in a holomorphic vector bundle.

Let $\pi: E \to M$ be a holomorphic vector bundle of rank r over a compact Hermitian manifold (M, J, g, ω). The $\overline{\partial}$ operator on each $\mathcal{A}^{p,q}(M)$ induces a operator $\overline{\partial}_E$ on each space $\mathcal{A}^{p,q}(E)$ in the following way. Let U be an open subset of M and let $\psi: \pi^{-1}(U) \to U \times \mathbb{C}^r$ be a trivialization of E over U. If $\{s_1, \dots, s_r\}$ is a smooth trivializating sections of U, i.e., each $s_i: U \to \pi^{-1}(U)$ and $\{s_1, \dots, s_r\}$ is local frame for $\pi^{-1}(U)$, then a section $\alpha \in \mathcal{A}^{p,q}(E)$ has local expression

$$\alpha = \sum_{j=1}^{r} \alpha_j \otimes s_j,$$

with $\alpha_j \in \mathcal{A}^{p,q}(M)$. The position

$$\overline{\partial}_E(\alpha) := \sum_{j=1}^{r} \overline{\partial}_E(\alpha_j) \otimes s_j$$

gives rise to a well defined and global operator on $\mathcal{A}^{p,q}(E)$ of type $(0,1)$ and such that $\overline{\partial}_E^2 = 0$. Then, it is possible to define the *Dolbeault cohomology of a holomorphic vector bundle* E as

$$H^{p,q}(M, E) := H^q(\mathcal{A}^{p,\bullet}(E), \overline{\partial}_E) = \frac{\mathrm{Ker}(\overline{\partial}_E: \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E))}{\mathrm{Im}(\overline{\partial}_E: \mathcal{A}^{p,q-1}(E) \to \mathcal{A}^{p,q}(E))}.$$

A *Hermitian structure* h on E is a hermitian scalar product h_x on each fiber E_x, for $x \in M$, which varies smoothly with respect to x. In particular, since $h \in E^* \otimes E^*$, it can be identified as a $\mathbb{C}$-antilinear isomorphism between E and E^*. Note, also, that a Hermitian structure h and a Hermitian metric g on (M, J) naturally induce a hermitian structure on every $\mathcal{A}^{p,q}(E)$ by the position, for every $\alpha_1 \otimes s_1, \alpha_2 \otimes s_2 \in \mathcal{A}^{p,q}(E)$,

$$\langle \alpha_1 \otimes s_1, \alpha_2 \otimes s_2 \rangle := g_H(\alpha_1, \alpha_2) \cdot h(s_1, s_2),$$

and then extending $\langle \cdot, \cdot \rangle$ by $\mathbb{C}$-linearity, respectively $\mathbb{C}$-antilinearity on the second component, to $\mathcal{A}^{p,q}(E)$.

For every $\eta \otimes s \in \mathcal{A}^{p,q}(E)$, the $\mathbb{C}$-*antilinear Hodge* $*$-*operator* on $\mathcal{A}^{p,q}(E)$ is the isomorphism given by

$$\begin{gathered} \overline{*}_E: \mathcal{A}^{p,q}(E) \to \mathcal{A}^{n-p,n-q}(E^*) \\ \eta \otimes s \mapsto \overline{*}_E(\eta \otimes s) := *\overline{\eta} \otimes h(s) \end{gathered}$$

and extended by linearity to $\mathcal{A}^{p,q}(E)$, where $*$ is the usual $\mathbb{C}$-linear Hodge $*$-operator on differential forms. Thus, $\overline{*}_E$ depends on the Hermitian structure h and on the Hermitian metric g on (M, J). As in the case of the usual $*$-operator, it holds that $\overline{*}_E \circ \overline{*}_{E^*} = (-1)^{p+q}$.

By means of $\overline{*}_E$, the adjoint operator of $\overline{\partial}_E$ with respect to the scalar product on each $\mathcal{A}^{p,q}(E)$

$$(\alpha, \beta) := \int_M \langle \alpha, \beta \rangle * 1, \qquad \text{for} \quad \alpha, \beta \in \mathcal{A}^{p,q}(E),$$

is defined as

$$\overline{\partial}_E^* := -\overline{*}_{E^*} \circ \overline{\partial}_E \circ \overline{*}_E$$

and the *Laplace operator on* $\mathcal{A}^{p,q}(E)$ is given by

$$\Delta_E := \overline{\partial}_E^* \overline{\partial}_E + \overline{\partial}_E \overline{\partial}_E^*.$$

It is clear that Δ_E is an elliptic, self-adjoint with respect to $(\cdot,\cdot)$, second order differential operator on each $\mathcal{A}^{p,q}(E)$.

A form $\alpha \in \mathcal{A}^{p,q}(E)$ is said to be Δ_E-*harmonic* if $\Delta_E(\alpha) = 0$. The space of Δ_E-harmonic forms will be denoted by $\mathcal{H}^{p,q}(M,E)$. Since Δ_E is elliptic, each space $\mathcal{H}^{p,q}(M,E)$ is finite dimensional and there exists a Hodge decomposition

$$\mathcal{A}^{p,q}(E) = \operatorname{Im} \overline{\partial}_E|_{\mathcal{A}^{p,q-1}(E)} \ \oplus\ \mathcal{H}^{p,q}(M,E) \ \oplus\ \operatorname{Im} \overline{\partial}_E^*|_{\mathcal{A}^{p,q+1}(E)}.$$

and the isomorphism of vector spaces

$$\mathcal{H}^{p,q}(M,E) \xrightarrow{\sim} H^{p,q}(M,E).$$

1.4 Complex formalities

Let (M,J) be a complex manifold. The study of the de Rham complex $(\mathcal{A}^\bullet(M), d)$ yields many interesting insight on the homotopy type of the manifold M. In particular, the manifold M is said to be *formal according to Sullivan*, if the de Rham complex $(\mathcal{A}^\bullet, d)$ has a fairly simple model; as a consequence, every triple Massey product vanishes. Moreover, a stronger notion of formality, involving Riemannian metric structures has been introduced by Kotschick in [87]; we recall the previous definitions in Appendix B.

Exploiting the complex structure, the $\mathcal{C}^\infty(M)$-algebra of (p,q)-forms $\mathcal{A}^{\bullet,\bullet}(M)$ on M endowed with the differential operators ∂ and $\overline{\partial}$ has a structure of *bidifferential bigraded algebra*. The study of such an object, started by Neisendorfer and Taylor, yields important information regarding holomorphic, in particular enables to study the notion of "holomorphic homotopy", see [106].

This section is devoted to first recalling the main definition in the more general context of bidifferential bigraded algebras and differential bigraded algebras and then adapting them to the setting of complex manifolds.

Definition 1.4.1. A *bigraded bidifferential algebra*, (shortly, *BBA*), is a triple $(\mathcal{A}, \partial_{\mathcal{A}}, \overline{\partial}_{\mathcal{A}})$ where $\mathcal{A} = \oplus_{i,j} \mathcal{A}_{i,j}$ is a bigraded algebra with a graded-commutative product, i.e., $\alpha \cdot \beta = (-1)^{\deg \alpha} \beta \wedge \alpha$, for every $\alpha, \beta \in \mathcal{A}$, and $\partial_{\mathcal{A}}$ and $\overline{\partial}_{\mathcal{A}}$ are morphisms of $\mathcal{A}$ of type $(1,0)$, respectively $(0,1)$, with respect to the bigradation of $\mathcal{A}$, such that

1. $\partial_{\mathcal{A}}$ and $\overline{\partial}_{\mathcal{A}}$ satisfy the Leibnitz rule, i.e., $\partial_{\mathcal{A}}(\alpha \cdot \beta) = \partial_{\mathcal{A}} \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \partial_{\mathcal{A}} \beta$, for every α, $\beta \in \mathcal{A}$ (analogously for $\overline{\partial}_{\mathcal{A}}$),
2. $\partial_{\mathcal{A}}$ and $\overline{\partial}_{\mathcal{A}}$ are differentials, i.e., $\partial_{\mathcal{A}}^2 = \overline{\partial}_{\mathcal{A}}^2 = 0$,
3. $\partial_{\mathcal{A}}$ and $\overline{\partial}_{\mathcal{A}}$ anticommute, i.e. $\partial_{\mathcal{A}} \overline{\partial}_{\mathcal{A}} = -\overline{\partial}_{\mathcal{A}} \partial_{\mathcal{A}}$.

A *morphism of BBA's* between two BBA's $(\mathcal{A}, \partial_{\mathcal{A}}, \overline{\partial}_{\mathcal{A}})$ and $(\mathcal{B}, \partial_{\mathcal{B}}, \overline{\partial}_{\mathcal{B}})$ is a map $f\colon \mathcal{A} \to \mathcal{B}$ which is morphism of algebras, preserves the bigradation, and commutes with the differentials of both BBA's. We say that two BBA's are *isomorphic* if there exists a bijective BBA's morphism between them.

The *cohomology* of a BBA $(\mathcal{A}, \partial_{\mathcal{A}}, \overline{\partial}_{\mathcal{A}})$ is the BBA $(H^{\bullet,\bullet}_{\overline{\partial}_{\mathcal{A}}}(\mathcal{A}), \partial_{\mathcal{A}}, 0)$ defined by

$$H^{p,q}_{\overline{\partial}_{\mathcal{A}}}(\mathcal{A}) := \frac{\operatorname{Ker}(\overline{\partial}_{\mathcal{A}}\colon \mathcal{A}_{p,q} \to \mathcal{A}_{p,q+1})}{\operatorname{Im}(\overline{\partial}_{\mathcal{A}}\colon \mathcal{A}_{p,q-1} \to \mathcal{A}_{p,q})}. \tag{1.4.1}$$

The product of elements of $H_{\overline{\partial}_{\mathcal{A}}}$ is given by $[\alpha]\cdot[\beta] \coloneqq [\alpha\cdot\beta]$, for every $[\alpha],[\beta] \in H_{\overline{\partial}_{\mathcal{A}}}(\mathcal{A})$.

Note that if $f\colon \mathcal{A}\to\mathcal{B}$ is morphism of BBA's, it induces a map in cohomology $H_{\overline{\partial}}(f)$ given by

$$H_{\overline{\partial}}(f)\colon H_{\overline{\partial}_{\mathcal{A}}}(\mathcal{A}) \longrightarrow H_{\overline{\partial}_{\mathcal{B}}}(\mathcal{B})$$
$$H_{\overline{\partial}}(f)[\alpha]_{\mathcal{A}} \coloneqq [f(\alpha)]_{\mathcal{B}},$$

which is well defined, since by definition, f commutes with both $\overline{\partial}_{\mathcal{A}}$ and $\overline{\partial}_{\mathcal{B}}$. Moreover, a morphism of BBA's f is said to a be a *quasi-isomorphism* if $H_{\overline{\partial}}(f)$ is an isomorphism.

Definition 1.4.2. Two BBA's $(\mathcal{A},\partial_{\mathcal{A}},\overline{\partial}_{\mathcal{A}})$ and $(\mathcal{B},\partial_{\mathcal{B}},\overline{\partial}_{\mathcal{B}})$ are said to be *equivalent* if there exists a family of BBA's $\{(\mathcal{C}_i,\partial_{\mathcal{C}_i},\overline{\partial}_{\mathcal{C}_i})\}_{i=0}^{2k}$ such that $(\mathcal{C}_0,\partial_{\mathcal{C}_0},\overline{\partial}_{\mathcal{C}_0}) = (\mathcal{A},\partial_{\mathcal{A}},\overline{\partial}_{\mathcal{A}})$, $(\mathcal{C}_{2k},\partial_{\mathcal{C}_{2k}},\overline{\partial}_{\mathcal{C}_{2k}}) = (\mathcal{B},\partial_{\mathcal{B}},\overline{\partial}_{\mathcal{B}})$, and for every $j\in\{0,\dots,k-1\}$ there exist morphisms of BBA's f_j and g_j

$$\begin{tikzcd}
 & (\mathcal{C}_{2j+1},\partial_{\mathcal{C}_{2j+1}},\overline{\partial}_{\mathcal{C}_{2j+1}}) \arrow[dl, "f_j"'] \arrow[dr, "g_j"] & \\
(\mathcal{C}_{2j},\partial_{\mathcal{C}_{2j}},\overline{\partial}_{\mathcal{C}_{2j}}) & & (\mathcal{C}_{2j+2},\partial_{\mathcal{C}_{2j+2}},\overline{\partial}_{\mathcal{C}_{2j+2}})
\end{tikzcd}$$

such that f_j and g_j are quasi-isomorphisms of BBA's.

The same concepts can be adapted to a class of more general algebras. A *differential bigraded algebra* (shortly, *DBA*), is a couple $(\mathcal{A},\overline{\partial}_{\mathcal{A}})$, where $\mathcal{A} = \oplus_{i,j}\mathcal{A}_{i,j}$ is a bigraded algebra endowed with a map $\overline{\partial}_{\mathcal{A}}$ of type $(0,1)$ with respect to the bigradation such that $\overline{\partial}_{\mathcal{A}}$ is a differential and satisfies the Leibnitz rule. Clearly, any BBA $(\mathcal{A},\partial_{\mathcal{A}},\overline{\partial}_{\mathcal{A}})$ is a DBA, by "forgetting" the operator $\partial_{\mathcal{A}}$. Morphisms of DBA's are bigraded algebras morphisms which commute with the differentials and two DBA's are *isomorphic* if there exist a bijective morphism of DBA between them. The *cohomology* of a DBA $(\mathcal{A},\overline{\partial}_{\mathcal{A}})$ is a DBA $(H_{\overline{\partial}_{\mathcal{A}}},0)$, with each $H^{p,q}_{\overline{\partial}_{\mathcal{A}}}(\mathcal{A})$ defined as in (1.4.1), since the definition of cohomology of BBA's relies only on operator of type $(0,1)$. Any morphism f of DBA's induces a morphism $H_{\overline{\partial}}(f)$ in cohomology; a *quasi-isomorphism of DBA's* is a morphism f of DBA's such that $H_{\overline{\partial}}(f)$ is an isomorphism of DBA's. The notion of equivalence between DBA's is analogous to the one of BBA's.

Definition 1.4.3. Two DBA's $(\mathcal{A},\overline{\partial}_{\mathcal{A}})$ and $(\mathcal{B},\overline{\partial}_{\mathcal{B}})$ are said to be *equivalent* if there exists a family of DBA's $\{(\mathcal{C}_i,\overline{\partial}_{\mathcal{C}_i})\}_{i=0}^{2k}$ such that $(\mathcal{C}_0,\overline{\partial}_{\mathcal{C}_0}) = (\mathcal{A},\overline{\partial}_{\mathcal{A}})$, $(\mathcal{C}_{2k},\overline{\partial}_{\mathcal{C}_{2k}}) = (\mathcal{B},\overline{\partial}_{\mathcal{B}})$, and for every $j\in\{0,\dots,k-1\}$ there exist morphisms of DBA's f_j and g_j

$$\begin{tikzcd}
 & (\mathcal{C}_{2j+1},\overline{\partial}_{\mathcal{C}_{2j+1}}) \arrow[dl, "f_j"'] \arrow[dr, "g_j"] & \\
(\mathcal{C}_{2j},\overline{\partial}_{\mathcal{C}_{2j}}) & & (\mathcal{C}_{2j+2},\partial_{\mathcal{C}_{2j+2}},\overline{\partial}_{\mathcal{C}_{2j+2}})
\end{tikzcd}$$

such that f_j and g_j are quasi-isomorphisms of DBA's.

Now, let (M,J) be a complex manifold. As pointed out, the double complex $(\mathcal{A}^{\bullet,\bullet}(M),\partial,\overline{\partial})$ has a structure of BBA and $(\mathcal{A}^{\bullet,\bullet},\overline{\partial})$ has a structure of DBA.

Definition 1.4.4. The complex manifold (M,J) is said to be *Dolbeault formal* (respectively, *weakly Dolbeault formal*) if $(\mathcal{A}^{\bullet,\bullet}(M),\partial,\overline{\partial})$ (respectively, $(\mathcal{A}^{\bullet,\bullet}(M),\overline{\partial})$) is equivalent, as a BBA (respectively, DBA), to a BBA $(\mathcal{B},\partial_{\mathcal{B}},\overline{\partial}_{\mathcal{B}})$ (respectively DBA $(\mathcal{B},\overline{\partial}_{\mathcal{B}})$) with $\partial_{\mathcal{B}} = \overline{\partial}_{\mathcal{B}} = 0$ (respectively, $\overline{\partial}_{\mathcal{B}} = 0$).

Clearly, a Dolbeault formal manifold is also weakly Dolbeault manifold. Moreover, by [106, Theorem 8], a manifold satisfying the $\partial\overline{\partial}$-lemma is Dolbeault formal. One can define a natural adaptation of triple Massey products to the complex $(\mathcal{A}^{\bullet,\bullet}(M), \partial, \overline{\partial})$, see [150].

Definition 1.4.5. Let $[\alpha] \in H^{p,q}_{\overline{\partial}}(M)$, $[\beta] \in H^{r,s}_{\overline{\partial}}(M)$, and $[\gamma] \in H^{u,v}_{\overline{\partial}}(M)$ such that

$$[\alpha] \cup [\beta] = 0 \in H^{p+r,q+s}_{\overline{\partial}}(M), \qquad [\beta] \cup [\gamma] = 0 \in H^{r+u,s+v}_{\overline{\partial}}(M),$$

i.e., there exist $f_{\alpha\beta} \in \mathcal{A}^{p+r,q+s-1}(M)$, $f_{\beta\gamma} \in \mathcal{A}^{r+u,s+v-1}(M)$ such that

$$\alpha \wedge \beta = \overline{\partial} f_{\alpha\beta}, \qquad \beta \wedge \gamma = \overline{\partial} f_{\beta\gamma}.$$

Then, the *triple Dolbeault-Massey product* (shortly, *Dolbeault-Massey product* or $\overline{\partial}$-product) $\langle [\alpha], [\beta], [\gamma] \rangle_{\overline{\partial}}$ is the (well defined) coset

$$\langle [\alpha], [\beta], [\gamma] \rangle_{\overline{\partial}} := [\alpha \wedge f_{\beta\gamma} + (-1)^{p+q} f_{\alpha\beta} \wedge \gamma] + \mathcal{J} \in H^{p+r+u,q+s+v-1}_{\overline{\partial}}(M)/\mathcal{J},$$

where $\mathcal{J} := [\alpha] \cup H^{r+u,s+v-1}_{\overline{\partial}}(M) + [\gamma] \cup H^{p+r,q+s-1}_{\overline{\partial}}(M)$ is an ideal of $H^{p+r+u,q+s+v-1}_{\overline{\partial}}(M)$. Note that the Dolbeault-Massey product $\langle [\alpha], [\beta], [\gamma] \rangle_{\overline{\partial}}$ is independent of the choice of representatives α, β, γ and of primitives $f_{\alpha\beta}, f_{\beta\gamma}$.

If M and N are two complex manifolds and $f\colon (\mathcal{A}^{\bullet,\bullet}(M), \overline{\partial}) \to (\mathcal{A}^{\bullet,\bullet}(N), \overline{\partial})$ is a DBA's quasi isomorphism, then Dolbeault-Massey products are functorial, i.e.,

$$H_{\overline{\partial}}(f)\langle [\alpha], [\beta], [\gamma] \rangle_{\overline{\partial}} = \langle H_{\overline{\partial}}(f)[\alpha], H_{\overline{\partial}}(f)[\beta], H_{\overline{\partial}}(f)[\gamma] \rangle_{\overline{\partial}},$$

for every Dolbeault-Massey product $\langle [\alpha], [\beta], [\gamma] \rangle_{\overline{\partial}}$ on M. Hence, it is easy to see that on a weakly Dolbeault formal manifold, every Dolbeault-Massey product vanishes (see [150, Proposition 3.2]).

Let (M, J) be a complex manifold. Analogously to [87], as in [150] we have that following.

Definition 1.4.6. A Hermitian metric g on (M, J) is said to be *geometrically-Dolbeault formal* if $\mathcal{H}^{\bullet,\bullet}_{\Delta_{\overline{\partial}}}(M)$ has a structure of algebra induced by $\wedge$, i.e., if $\alpha \in \mathcal{H}^{p,q}_{\Delta_{\overline{\partial}}}(M)$, $\beta \in \mathcal{H}^{r,s}_{\Delta_{\overline{\partial}}}(M)$, then $\alpha \wedge \beta \in \mathcal{H}^{p+r,q+s}_{\Delta_{\overline{\partial}}}(M)$.

A complex manifold (M, J) endowed with a geometrically-Dolbeault formal metric is called a *geometrically-Dolbeault formal manifold.*

Proposition 1.4.7. *([150, Proposition 2.1, 2.2]). Let (M, J, g) be a geometrically-Dolbeault formal manifold. Then, (M, J) is weakly Dolbeault formal. Moreover, if $\mathcal{H}^{\bullet,\bullet}_{\Delta_{\overline{\partial}}}(M)$ is ∂-invariant, then (M, J) is Dolbeault formal.*

The notions of Dolbeault formality are then related in the following way

$$\begin{array}{ccccc} \text{Dolbeault formality} & \Longrightarrow & \text{Weak Dolbeault formality} & \Longrightarrow & \text{Vanishing of } \overline{\partial}\text{-products} \\ & & \Uparrow & & \\ & & \text{Geometric Dolbeault formality} & & \end{array} \tag{1.4.2}$$

and

$$\text{Geometric Dolbeault formality} \;+\; \partial\text{−invariance of } \mathcal{H}^{\bullet,\bullet}_{\Delta_{\overline{\partial}}}(M) \Longrightarrow \text{Dolbeault formality}. \tag{1.4.3}$$

Analogous notions of geometric formality and triple Massey products for Bott-Chern and Aeppli cohomology has been introduced by [18], (see also [147, 10]).

Definition 1.4.8. Let $[\alpha] \in H^{p,q}_{BC}(M), [\beta] \in H^{r,s}_{BC}(M), [\gamma] \in H^{u,v}_{BC}(M)$ such that

$$[\alpha] \cup [\beta] = 0 \in H^{p+r,q+s}_{BC}(M), \qquad [\beta] \cup [\gamma] = 0 \in H^{r+u,s+v}_{BC}(M)$$

i.e., there exist $f_{\alpha\beta} \in \mathcal{A}^{p+q-1,r+s-1}(M)$, $f_{\beta\gamma} \in \mathcal{A}^{r+u-1,s+v-1}(M)$ such that

$$(-1)^{p+q}\alpha \wedge \beta = \partial\overline{\partial} f_{\alpha\beta}, \qquad (-1)^{r+s}\beta \wedge \gamma = \partial\overline{\partial} f_{\beta\gamma}. \tag{1.4.4}$$

Then, the *triple Aeppli-Bott-Chern Massey product* (shortly, BC-product) $\langle[\alpha],[\beta],[\gamma]\rangle_{ABC}$ is the coset

$$\langle[\alpha],[\beta],[\gamma]\rangle_{ABC} := [(-1)^{p+q}\alpha \wedge f_{\beta\gamma} - (-1)^{r+s} f_{\alpha\beta} \wedge \gamma] \in H^{p+r+u-1,q+s+v-1}_{A}(M)/\mathcal{J},$$

with $\mathcal{J} := [\alpha] \cup H^{r+u-1,s+v-1}_{A}(M) + [\gamma] \cup H^{p+r-1,q+s-1}_{A}(M)$ ideal of $H^{p+r+u-1,q+s+v-1}_{A}(M)$. Notice that this definition is independent of choice of representatives α, β, γ and primitives $f_{\alpha\beta}, f_{\beta\gamma}$.

Definition 1.4.9. A Hermitian metric g on (M,J) is said to be *geometrically-Bott-Chern formal* if $\mathcal{H}^{\bullet,\bullet}_{\Delta_{BC}}(M)$ has a structure of algebra induced by $\wedge$, i.e., if $\alpha \in \mathcal{H}^{p,q}_{\Delta_{BC}}(M)$, $\beta \in \mathcal{H}^{r,s}_{\Delta_{BC}}(M)$, then $\alpha \wedge \beta \in \mathcal{H}^{p+r,q+s}_{\Delta_{BC}}(M)$.

A complex manifold admitting a geometrically-Bott-Chern formal metric is said to be a *geometrically-Bott-Chern formal manifold*.

These cohomological and metric notions are related as shown by the following proposition (see [147]).

Proposition 1.4.10. *Let M be a compact complex manifold. If M is geometrically-Bott-Chern formal then the Aeppli-Bott-Chern-Massey triple products are trivial.*

1.5 Special structures on complex manifolds

Let (M,J,g,ω) be a compact Hermitian manifold. The metric g is said to be *Kähler* if

$$d\omega = 0.$$

The existence of such metrics forces many strong topological and cohomological restraints on the manifold, as recalled in the introduction. As a consequence, there exist many natural classes of non-Kähler manifolds which admit metrics with weaker properties than Kähler metrics, e.g., compact quotients of nilpotent Lie groups by a discrete uniform subgroup which are not tori, see [24].

However, depending on the closedness of the fundamental form ω (or its powers) with respect to certain differential operators, many special metric structures which generalize the Kähler condition arise.

Definition 1.5.1. ([61, 26, 78, 103]) A Hermitian metric (g,ω) on a complex manifold (M,J) of complex dimension n is said to be

- *Gauduchon*, or *regular*, if
$$\partial\overline{\partial}\omega^{n-1} = 0,$$
- *Strong Kähler with torsion* (shortly, *SKT*), or *pluriclosed*, if
$$\partial\overline{\partial}\omega = 0,$$
or, recalling (1.2.5),
$$dd^c\omega = 0.$$

- *astheno-Kähler* if
$$\partial\overline{\partial}\omega^{n-2} = 0,$$
- *balanced*, or *co-closed*, if
$$d\omega^{n-1} = 0.$$
Since $\omega = \overline{\omega}$ and $d\omega^{n-1} = \partial\omega^{n-1} + \overline{\partial}\omega^{n-1} = \partial\omega^{n-1} + \overline{(\partial\omega^{n-1})} = \overline{(\overline{\partial}\omega^{n-1})} + \overline{\partial}\omega^{n-1}$, the metric g is balanced if, and only if, $\partial\omega^{n-1} = 0$, if, and only if, $\overline{\partial}\omega^{n-1} = 0$.

By definition, for certain complex dimensions these metric notions coincide, i.e., for $n = 2$, Gauduchon and SKT metrics and Kähler and balanced metrics, for $n = 3$, SKT and astheno-Kähler metrics; however, in higher dimensions there exists many examples of manifolds in which the notions are well distinct ([125]).

Note that the fundamental forms ω of Kähler, Gauduchon, SKT, astheno-Kähler, and balanced metrics are particular cases of more geneal special structures on the complex manifold (M, J). In order to recall the definition of such structures, let us fix (V, J) a $2n$-dimensional real vector space endowed with an almost complex structure J. In the following, we will consider the spaces of *real* (p,p)*-covectors* on V, i.e., elements of the spaces

$$\bigwedge_{\mathbb{R}}^{p,p} V := \{\psi \in \bigwedge^{p,p} V \mid \psi = \overline{\psi}\}, \qquad 1 \le p \le n,$$

where the space of (p,q)-covectors $\bigwedge^{p,q} V$ is given, as usual, by (1.1.5). Let us fix the constant $\sigma_p := i^{p^2} 2^{-p}$. It is easy to check that, for every $(p,0)$-covector $\psi \in \bigwedge^{p,0} V$, it holds

$$\overline{\sigma_p \psi \wedge \overline{\psi}} = \sigma_p \psi \wedge \overline{\psi},$$

i.e., the (p,p)-covector $\sigma_p \psi \wedge \overline{\psi} \in \bigwedge_{\mathbb{R}}^{p,p} V$. Therefore, if $\{\eta^1, \dots, \eta^n\}$ is base for $(V^*)^{1,0}$, the set

$$\{\sigma_p \eta^{i_1} \wedge \cdots \wedge \eta^{i_p} \wedge \overline{\eta^{i_1}} \wedge \cdots \wedge \overline{\eta^{i_p}} \mid 1 \le i_1 < \cdots < i_p \le n\}$$

forms a base of $\bigwedge_{\mathbb{R}}^{p,p} V$. By definition of the extension of J to k-covectors (1.1.6), it is clear that every real (p,p)-covector ψ is J-invariant, i.e., $J\psi = \psi$.

With these notation, the (n,n)-covector on V defined by

$$\mathrm{Vol} := \left(\frac{i}{2}\eta^1 \wedge \overline{\eta^1}\right) \wedge \cdots \wedge \left(\frac{i}{2}\eta^n \wedge \overline{\eta^n}\right);$$

is then

$$\mathrm{Vol} = \sigma_n\, \eta^1 \wedge \cdots \wedge \eta^n \wedge \overline{\eta^1} \wedge \cdots \wedge \overline{\eta^n},$$

i.e., the covector Vol is a volume form on V.

Definition 1.5.2 ([5])**.** A covector $\psi \in \bigwedge^{p,0} V$ is said to be *simple* or *decomposable* if

$$\psi = \alpha^1 \wedge \cdots \wedge \alpha^p,$$

for suitable $\alpha^1, \dots, \alpha^p \in V^{1,0}$.

Definition 1.5.3. A real (n,n)-covector $\psi \in \bigwedge_{\mathbb{R}}^{n,n} V$ is said to be *positive*, respectively *strictly positive*, if

$$\psi = a\mathrm{Vol},$$

where $a \ge 0$, respectively, $a > 0$.

Definition 1.5.4. A real (p,p)-covector $\Omega \in \bigwedge_{\mathbb{R}}^{p,p} V$ is said to be *weakly positive* if given any non-zero simple $(n-p)$-covector η, the real (n,n)-covector

$$\Omega \wedge \sigma_{n-p}\eta \wedge \overline{\eta}$$

is positive.

Definition 1.5.5. A real (p,p)-covector Ω is said to be *transverse* if, given any non-zero simple $(n-p)$-covector η, the real (n,n)-covector

$$\Omega \wedge \sigma_{n-p}\eta \wedge \overline{\eta}$$

is strictly positive.

Let now (M,J) be a (almost) complex manifold of real dimension $2n$ and denote by

$$A_{\mathbb{R}}^{p,p}(M) := \{\psi \in A^{p,q}(M) \mid \psi = \overline{\psi}\}$$

the space of *real* (p,p)*-forms*. Fix $1 \le p \le 2n$.

Definition 1.5.6 ([142])**.** A *p-Kähler form* on (M,J) is a real d-closed transverse (p,p)-form Ω, that is $d\Omega = 0$, and, at every $x \in M$, $\Omega_x \in \bigwedge_{\mathbb{R}}^{p,p}(T_x^*M)$ is transverse. The triple (M,J,Ω) s said to be an *(almost) p-Kähler manifold.*

Definition 1.5.7 ([5])**.** A *p-pluriclosed form* on (M,J) is a real dd^c-closed transverse (p,p)-form Ω, that is Ω is dd^c-closed and, at every $x \in M$, $\Omega_x \in \bigwedge_{\mathbb{R}}^{p,p}(T_x^*M)$ is transverse. The triple (M,J,Ω) is said to be an *(almost) p-pluriclosed manifold.*

In general, a p-pluriclosed manifold (M,J,Ω) does not admit an Hermitian metric g with fundamental form ω such that $\partial\overline{\partial}\omega^p = 0$. However, this is the case for $p=1$, for which the existence of a 1-pluriclosed form on (M,J) implies the existence of a SKT metric on (M,J).

Viceversa, if (M,J) admits an astheno-Kähler metric g (respectively SKT metric) with fundamental associated form ω, then (M,J,ω^{n-2}) (respectively (M,J,ω)) is an $(n-2)$-pluriclosed (respectively 1-pluriclosed) manifold.

1.6 Invariant complex structures and cohomology of nilmanifolds

We refer to Appendix C for a basic introduction on left-invariant vector fields and differential forms and the computation of the de Rham cohomology via the complex of left-invariant forms on nilmanifolds and completely solvable solvmanifolds.

In this section, we recall the main notion of invariant complex structures on nilmanifolds and a way to compute the Dolbeault cohomology (1.3.2), the Bott-Chern cohomology (1.3.1), and the Aeppli cohomology (1.3.3), of such manifolds via the corresponding left-invariant cohomology.

Let G be a real $2n$-dimensional Lie.

Definition 1.6.1. An almost complex structure on G is said to be *left-invariant* if, for every $a,x \in G$, on $T_{ax}G$ it holds that

$$(\hat{L_a})_x J_x = J_{ax},$$

where

$$(\hat{L_a})_x J_x := (dL_a)_x \circ J_x \circ (dL_{a^{-1}})_{ax},$$

with dL_a the differential of the left translation map L_a (see Appendix C).

Let $\mathfrak{g} \simeq T_eG$ be the associated Lie algebra of the Lie group G.

Definition 1.6.2. An *almost complex structure* on $\mathfrak{g}$ is an almost complex structure on $\mathfrak{g}$ considered as a vector space (see section 1.1).

Note that for every almost complex structure J on $\mathfrak{g}$, there corresponds a unique left-invariant almost complex structure $\hat{J}$ on G such that $\hat{J}_e = J$, and it holds $\hat{J}_x = (\hat{L_x})_e J$, for every $a \in G$. Hence, a left-invariant almost complex structure $\hat{J}$ on G can be induced by assigning an almost complex structure J on $\mathfrak{g}$. Moreover, by Newlander-Niremberg theorem (see section 1.2) such an induced almost complex structure $\hat{J}$ is integrable on G if, and only if

$$N_J(X,Y) = [JX,JY] - J[X,JY] - J[JX,Y] - [X,Y] = 0, \quad \forall X,Y \in \mathfrak{g},$$

i.e., if and only if, the Nijenhuis tensor N_J identically vanishes on $\mathfrak{g}\times\mathfrak{g}$. Consequently, we now work on $\mathfrak{g}$. The extension of J to the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ induces the the usual decompositions

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1},$$

where $\mathfrak{g}^{1,0} := \{X - iJX : X \in \mathfrak{g}_{\mathbb{C}}\}$ and $\mathfrak{g}^{0,1} := \{X + iJX : X \in \mathfrak{g}_{\mathbb{C}}\}$, and

$$\mathfrak{g}^*_{\mathbb{C}} = (\mathfrak{g}^{1,0})^* \oplus (\mathfrak{g}^{0,1})^*,$$

where $(\mathfrak{g}^{1,0})^* = \{\alpha + iJ\alpha : \alpha \in \mathfrak{g}^*_{\mathbb{C}}\}$ and $(\mathfrak{g}^{0,1})^* = \{\alpha - iJ\alpha : \alpha \in \mathfrak{g}^*_{\mathbb{C}}\}$. On the exterior powers $\wedge^k(\mathfrak{g}^*_{\mathbb{C}})$, the following decompositions hold

$$\wedge^k(\mathfrak{g}^*_C) = \bigoplus_{p+q=k} \wedge^{p,q}\mathfrak{g}, \qquad \wedge^{p,q}\mathfrak{g} := \wedge^p(\mathfrak{g}^{1,0})^* \otimes \wedge^q(\mathfrak{g}^{0,1})^*.$$

Let $\{Z_1,\dots,Z_n\}$ now be a $\mathbb{C}$-base of $\mathfrak{g}^{1,0}$. Then, as recalled in section 1.2, the vanishing of J is equivalent to

$$[Z_i,Z_j] = c^k_{ij}Z_k, \quad c^k_{ij} \in \mathbb{C}, \quad \forall i,j \in \{1,\dots,n\},$$

or equivalently,

$$[\overline{Z_i},\overline{Z_j}] = c^{\overline{k}}_{\overline{ij}}\overline{Z_k}, \quad c^{\overline{k}}_{\overline{ij}} \in \mathbb{C}, \quad \forall i,j \in \{1,\dots,n\},$$

i.e., $[\mathfrak{g}^{1,0},\mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$, or equivalently, $[\mathfrak{g}^{0,1},\mathfrak{g}^{0,1}] \subset \mathfrak{g}^{0,1}$. The complex numbers

$$\{c^k_{ij} : \quad i,j,k \in \{1,\dots,n,\overline{1}\dots,\overline{n}\}\} \tag{1.6.1}$$

are called the *complex structure constants*, and they completely determine the complex structure J on $\mathfrak{g}$, hence, the complex structure $\hat{J}$ on G. For the sake of completeness, we write the commutators

$$\begin{aligned}
[Z_i,Z_j] &= c^k_{ij}Z_k, & i,j \in \{1,\dots,n\}\\
[Z_i,\overline{Z_j}] &= c^k_{i\overline{j}}Z_k + c^{\overline{k}}_{i\overline{j}}\overline{Z_k}, & i,j \in \{1,\dots,n\}\\
[\overline{Z_i},Z_j] &= c^k_{\overline{i}j}Z_k + c^{\overline{k}}_{\overline{i}j}\overline{Z_k}, & i,j \in \{1,\dots,n\},\\
[\overline{Z_i},\overline{Z_j}] &= c^{\overline{k}}_{\overline{ij}}\overline{Z_k}, & i,j \in \{1,\dots,n\}.
\end{aligned}$$

Since the bracket $[\cdot,\cdot]$ is skew-symmetric, the following relations hold

$$c^k_{ii} = 0, \qquad c^{\overline{k}}_{\overline{ii}} = 0, \qquad c^k_{ij} = -c^k_{ji}, \qquad c^k_{i\overline{j}} = -c^k_{\overline{j}i}, \qquad c^{\overline{k}}_{i\overline{j}} = -c^{\overline{k}}_{\overline{j}i}, \qquad c^{\overline{k}}_{\overline{ij}} = -c^{\overline{k}}_{\overline{ji}}.$$

Moreover, by complex conjugation, it holds that

$$c_{\bar{i}\bar{j}}^{\bar{k}} = \overline{c_{ij}^{k}}, \qquad c_{\bar{i}j}^{k} = -\overline{c_{i\bar{j}}^{\bar{k}}}, \qquad c_{\bar{i}j}^{\bar{k}} = -\overline{c_{i\bar{j}}^{k}}.$$

Therefore, the complex structure J is actually determined by the commutators

$$\begin{aligned} [Z_i, Z_j] &= c_{ij}^{k} Z_k \qquad i < j \\ [Z_i, \overline{Z_j}] &= c_{i\bar{j}}^{k} Z_k + c_{i\bar{j}}^{\bar{k}} \overline{Z_k}, \qquad i \leq j. \end{aligned}$$

Let then $\{\eta^1, \dots, \eta^n\} \subset (\mathfrak{g}^{1,0})^*$ be the dual base of $\{Z_1, \dots, Z_n\}$. By the relation between left-invariant vector fields and left-invariant forms

$$d\alpha(X, Y) = -\alpha([X, Y]),$$

we obtain the *complex structure equations*

$$d\eta^k = -\sum_{i<j} c_{ij}^{k} \eta^{ij} - \sum_{i \leq j} c_{i\bar{j}}^{k} \eta^{i\bar{j}} + \sum_{i<j} \overline{c_{i\bar{j}}^{\bar{k}}} \eta^{j\bar{i}}, \qquad k \in \{1, \dots, n\}, \tag{1.6.2}$$

or, by making use of the splitting $d = \partial + \overline{\partial}$,

$$\begin{cases} \partial\eta^k = -\sum_{i<j} c_{ij}^{k} \eta^{ij}, \qquad k \in \{1, \dots, n\}, \\ \overline{\partial}\eta^k = -\sum_{i \leq j} c_{i\bar{j}}^{k} \eta^{i\bar{j}} + \sum_{i<j} \overline{c_{i\bar{j}}^{\bar{k}}} \eta^{j\bar{i}}, \qquad k \in \{1, \dots, n\} \end{cases}.$$

Special properties of a left-invariant complex structure J on a Lie group G are reflected on the structure constants $c_{ij}^{k}, c_{i\bar{j}}^{k}, c_{i\bar{j}}^{\bar{k}}$ with respect to a fixed base of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}^*$:

- J is *holomorphically parallelizable* if the holomorphic tangent bundle $(T^{1,0}G)^*$ is holomorphically trivial, i.e., there exist a global coframe of holomorphic $(1,0)$-forms on G. This holds if, and only if, there exists a coframe $\{\eta^1, \dots, \eta^n\}$ of (left-invariant) $(1,0)$-forms such that

$$c_{i\bar{j}}^{k} = c_{i\bar{j}}^{\bar{k}} = 0.$$

- J is *abelian* if $[JX, JW] = [X, W]$ for every $X, W \in \mathfrak{g}$. This implies that, for every coframe $\{\eta^1, \dots, \eta^n\}$ of $(\mathfrak{g}^{1,0})^*$, it holds

$$c_{ij}^{k} = 0.$$

- J is *nilpotent* if the ascending series $\{\mathfrak{g}_l^J\}_{l \geq 0}$ defined by

$$\mathfrak{g}_0^J = \{0\}, \qquad \mathfrak{g}_l^J := \{X \in \mathfrak{g} | [X, \mathfrak{g}] \subseteq \mathfrak{g}_{l-1}^J, [JX, \mathfrak{g}] \subseteq \mathfrak{g}_{l-1}^J\}$$

satisfies $\mathfrak{g}_{k_0}^J = \mathfrak{g}$, for some $k_0 > 0$. This is equivalent ([45, Theorem 2], see also [44, Theorem 9]) to the existence of a $\mathbb{C}$-coframe $\{\eta^1, \dots, \eta^n\}$ such that, for every k, if either $i \geq k$ or $j \geq k$ or $j \geq i$, then $c_{ij}^{k} = 0$ and if either $i \geq k$ or $j \geq k$, then $c_{i\bar{j}}^{k} = c_{i\bar{j}}^{\bar{k}} = 0$, i.e., for every k,

$$d\eta^k = \sum_{i<j<k} -c_{ij}^{k} \eta^{ij} - \sum_{i,j<k} c_{i\bar{j}}^{k} \eta^{i\bar{j}} + \sum_{i,j<k} \overline{c_{i\bar{j}}^{\bar{k}}} \eta^{j\bar{i}}.$$

- J is *rational* if $\mathfrak{g}$ admits a *rational structure*, i.e., a vector space $\mathfrak{h}$ over $\mathbb{Q}$ such that $\mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{R} = \mathfrak{g}$, and it holds that $J(\mathfrak{h}) \subset \mathfrak{h}$. In particular, this implies the existence of a coframe $\{\eta^1, \dots, \eta^n\}$ of $(\mathfrak{g}^{1,0})^*$ such that

$$c_{ij}^{k}, c_{i\bar{j}}^{k}, c_{i\bar{j}}^{\bar{k}} \in \mathbb{Q}[i].$$

Let now $M := \Gamma\backslash G$ be a real $2n$-dimensional nilmanifold, i.e., the compact quotient of a nilpotent simply connected Lie group G by a discrete uniform subgroup Γ (see definition in Appendix C). Any left-invariant tensor on G is also Γ-invariant, hence it descends to a well defined object on the quotient M. We will call *invariant* a tensor on M which pulls back to a left-invariant one on the G.

Let then J be an invariant integrable almost complex structure on M. If $\mathfrak{g}$ is the Lie algebra associated to G, let us denote the complex of left-invariant forms endowed with the ∂ and $\overline{\partial}$ differentials $(\bigwedge^{\bullet,\bullet}\mathfrak{g}, \partial, \overline{\partial})$. Then, the *invariant Dolbeault cohomology*, *invariant Bott-Chern cohomology*, and the *invariant Aeppli cohomology* of M are defined, respectively, as the spaces

$$H^{p,q}_{\overline{\partial}}(\mathfrak{g}) := \frac{\operatorname{Ker}(\overline{\partial}\colon \bigwedge^{p,q}\mathfrak{g} \to \bigwedge^{p,q+1}\mathfrak{g})}{\operatorname{Im}(\overline{\partial}\colon \bigwedge^{p,q-1}\mathfrak{g} \to \bigwedge^{p,q}\mathfrak{g})}$$

$$H^{p,q}_{BC}(\mathfrak{g}) := \frac{\operatorname{Ker}(\partial\colon \bigwedge^{p,q}\mathfrak{g} \to \bigwedge^{p+1,q}\mathfrak{g}) \cap \operatorname{Ker}(\overline{\partial}\colon \bigwedge^{p,q}\mathfrak{g} \to \bigwedge^{p,q+1}\mathfrak{g})}{\operatorname{Im}(\partial\overline{\partial}\colon \bigwedge^{p-1,q-1}\mathfrak{g} \to \bigwedge^{p,q}\mathfrak{g})}$$

$$H^{p,q}_{A}(\mathfrak{g}) := \frac{\operatorname{Ker}(\partial\overline{\partial}\colon \bigwedge^{p,q}\mathfrak{g} \to \bigwedge^{p+1,q+1}\mathfrak{g})}{\operatorname{Im}(\partial\colon \bigwedge^{p-1,q}\mathfrak{g} \to \bigwedge^{p,q}\mathfrak{g}) + \operatorname{Im}(\overline{\partial}\colon \bigwedge^{p,q-1}\mathfrak{g} \to \bigwedge^{p,q}\mathfrak{g})}$$

As in the differentiable case, conditions on the complex structure under which the inclusion

$$\bigwedge^{\bullet,\bullet}\mathfrak{g} \hookrightarrow \mathcal{A}^{\bullet,\bullet}(M)$$

induces an isomorphism in cohomology

$$H^{\bullet,\bullet}_{\sharp}(\mathfrak{g}) \simeq H^{\bullet,\bullet}_{\sharp}(M), \qquad \sharp \in \{\overline{\partial}, BC, A\}$$

have been estabilished. The following theorem by Angella in [11] summarizes such conditions.

Theorem 1.6.3. *Let $(M = \Gamma\backslash G, J)$ be a $2n$-dimensional nilmanifold endowed with a invariant complex structure J and let $\mathfrak{g}$ be the associated Lie algebra of G. If one the following conditions is satisfied*

- *(M, J) is holomorphically parallelizable,*
- *J is abelian,*
- *J is nilpotent,*
- *J is a rational complex structure,*

then the inclusion

$$\bigwedge^{\bullet,\bullet}\mathfrak{g} \hookrightarrow \mathcal{A}^{\bullet,\bullet}(M)$$

induces an isomorphism in cohomology

$$H^{\bullet,\bullet}_{\sharp}(\mathfrak{g}) \simeq H^{\bullet,\bullet}_{\sharp}(M), \qquad \sharp \in \{\overline{\partial}, BC, A\}.$$

Remark 1.6.4. In order to make computations in complex cohomology via the previous theorem, one needs to dispose of a real $2n$-dimensional simply connected nilpotent Lie group G which admits a coframe with rational structure constants and which admits a left-invariant integrable almost structure satisying one of the conditions of Theorem 1.6.3. One way to do this, is to fix a base of $(1,0)$-forms $\{\eta^1, \ldots, \eta^n\}$ of a formal Lie algebra $\mathfrak{g}$ and assign complex structure equations (1.6.2) such that J is both nilpotent and rational. Then, the dual $\mathfrak{g}^*$ of the real Lie algebra $\mathfrak{g}$ underlying $\mathfrak{g}_{\mathbb{C}}$ is determined by $\mathfrak{g}^* = \operatorname{Span}_{\mathbb{R}}\langle e^j\rangle$, where $e^{2j-1} := \mathfrak{Re}(\eta^j)$ and $e^{2j} := \mathfrak{Im}(\eta^j)$, $j \in \{1, \ldots, n\}$.

Consequently, the real constant structure of $\mathfrak{g}$ will be determined by the expressions of $d(e^j)$ in terms of the base $\{e^j\}_{j=1}^{2n}$ of $\mathfrak{g}^*$. By passing to dual, the Lie algebra $\mathfrak{g}$ is then determined; in particular, it turns out that $\mathfrak{g}$ is a real nilpotent Lie algebra. By a classical theorem in Lie group theory, there exists a unique simply connected nilpotent Lie group G (up to isomorphism) such that its Lie associated Lie algebra is precisely $\mathfrak{g}$. The Lie group G is then endowed with a left-invariant integrable almost complex structure J such that $\mathfrak{g}_{\mathbb{C}}^* = (\mathfrak{g}^{1,0})^* \oplus (\mathfrak{g}^{0,1})^*$, where the set $\{\eta^j := e^{2j-1} + ie^{2j}\}_{j=1}^n$ is a basis for $(\mathfrak{g}^{1,0})^*$. Moreover, since the structure constant of the coframe $\{e^j\}_{j=1}^{2n}$ are rational, by Mal'cev theorem, the Lie group G admits a discrete uniform subgroup Γ such that $M = \Gamma\backslash G$ is a nilmanifold. The left-invariant almost complex structure J descends to a invariant one on M, so that (M, J) is a nilmanifold endowed with a nilpotent and rational invariant integrable almost complex structure; hence, Theorem 1.6.3 applies.

1.7 Deformations of complex structures

In this section, we recall the fundamental definitions and results of deformation theory of complex manifolds. Let B be a domain of $\mathbb{R}^m$ (respectively, $\mathbb{C}^m$) and $\{M_t\}_{t\in B}$ a family of compact complex manifolds.

Definition 1.7.1. We say that M_t *depends differentiably* (respectively, *holomorphically*) on $t \in B$, or, equivalently, that $\{M_t\}_{t\in B}$ *forms a differentiable* (respectively, *holomorphic*) *family* if there exists a differentiable (respectively, complex) manifold $\mathcal{M}$ and a differentiable (respectively, holomorphic) proper map $\pi\colon \mathcal{M} \to B$ such that

1. $\pi^{-1}(t) = M_t$ as a complex manifold for every $t \in B$,
2. the rank of the Jacobian of π is equal to the dimension (respectively, complex dimension) of B at each point of $\mathcal{M}$.

From 2. of the definition that every M_t, for $t \in B$, is a submanifold (respectively, complex submanifold) of $\mathcal{M}$. In what follows, we will denote also by $(\mathcal{M}, \pi, B)$ the differentiable (respectively, holomorphic) family $\{M_t\}_{t\in B}$.

Definition 1.7.2. If M, N are compact complex manifolds, we say that M *is a differentiable* (respectively, *holomorphic*) *deformation of* N if there exists a differentiable (respectively, holomorphic) family $\{M_t\}_{t\in B}$ over a domain B of $\mathbb{R}^m$ (respectively, $\mathbb{C}^m$), with $M_{t_0} = M$, $M_{t_1} = N$ for some $t_0, t_1 \in B$.

A classical theorem by Ehresmann, see [49] or [75, Proposition 6.2.2], shows that if $\{M_t\}_{t\in B}$ is a differentiable family of complex manifolds, then M_{t_1} and M_{t_2} are diffeomorphic as differentiable manifolds, for any $t_1, t_2 \in B$. Hence, from the differentiable point of view, it holds

$$\mathcal{M} \simeq M_{t_0} \times B, \tag{1.7.1}$$

i.e., the manifold $\mathcal{M}$ can be regarded as the product of a fixed M_{t_0}, for $t_0 \in B$, and the base manifold B. The complex manifold M_{t_0} is usually called the *central fiber*. From the complex point of view, each fiber M_t, $t \in B$, can be seen as the complex manifold (M_{t_0}, J_t), where M_{t_0} denotes the underlying differentiable structure of the central fiber and J_t is an integrable almost complex structure on M_{t_0} which varies smoothly with $t \in B$.

Definition 1.7.3. A property P depending on the complex structure of a complex manifold (M, J) is said to be

- *open under holomorphic (respectively, differentiable) deformations* if for every holomorphic (respectively, differentiable) family $(M, J_t)_{t \in B}$ such that $(M, J_{t_0}) = (M, J)$, if P holds for (M, J), then P holds for every (M, J_t), for $t \in B$;
- *closed under holomorphic (respectively, differentiable) deformations* if for every holomorphic (respectively, differentiable) family $(M, J_t)_{t \in B}$ such that $(M, J_{t_0}) = (M, J)$, if P holds for every (M, J_t), for $t \in B$, then P holds also on (M, J).

Let $(\mathcal{M}, \pi, B)$ be a differentiable family of compact n-dimensional complex manifolds over B. For the sake of simplicity, we assume $t_0 = 0$ and $B = B(0,1) \subset \mathbb{R}^m$, i.e. $B = \{t \in \mathbb{R}^m : |t| < 1\}$.

Let us consider a system of local coordinates $\{\mathcal{U}_j, (\zeta_j, t)\}$ of $\mathcal{M}$ such that each $\mathcal{U}_j$ can be identified with the set

$$\{(\zeta_j, t) : |\zeta_j| < 1, |t| < 1\} \subset \mathbb{C}^n \times \mathbb{R}^m, \quad \pi(\zeta_j, t) = t,$$

with transition functions f_{jk}, which identify points in $\mathcal{U}_j \cap \mathcal{U}_k \neq \varnothing$ by

$$\zeta_j = f_{jk}(\zeta_k, t),$$

and which are differentiable with respect to (ζ_k, t) and are holomorphic with respect to ζ_k for any fixed t. We note that each $\mathcal{U}_j \simeq U_j \times B$, where $U_j = \{\zeta_j : |\zeta_j| < 1\}$.

By (1.7.1), we can describe the local coordinates of $\mathcal{U}_j$ as differentiable functions of coordinates of $M_0 = \pi^{-1}(0)$, that is,

$$\zeta_j = \zeta_j(z, t), \tag{1.7.2}$$

where z are local holomorphic coordinates on M_0 which at this moment we consider as differentiable coordinates. In particular, the coordinates $\zeta_j(z,t)$ are differentiable functions of (z,t), whereas they depend holomorphically on z for each fixed value of t.

With the aid of the expressions (1.7.2), we can actually describe the complex structure on each M_t, $t \in B$, via a smooth $(0,1)$-vector form $\varphi(t) \in \mathcal{A}^{0,1}(T^{1,0}M_0)$, defined starting from the local transition functions f_{jk} (see [85, page 150]).

In fact, since both $\{\zeta_j^1(z,0), \ldots, \zeta_j^n(z,0)\}$ and $\{z^1, \ldots, z^n\}$ are local holomorphic coordinates on M_0,

$$\det \left(\frac{\partial \zeta_j^\alpha(z,0)}{\partial z^\lambda} \right)_\alpha^\lambda \neq 0.$$

Therefore, in a small neighborhood of $0 \in \mathbb{R}^m$, it holds that

$$\det \left(\frac{\partial \zeta_j^\alpha(z,t)}{\partial z^\lambda} \right)_\alpha^\lambda \neq 0.$$

Set $A_{j\alpha}^\lambda := \left(\left(\frac{\partial \zeta_j^\alpha(z,t)}{\partial z^\lambda} \right)_\alpha^\lambda \right)^{-1}$ and

$$\varphi_j^\lambda(z,t) := \sum_{\alpha=1}^n A_{j\alpha}^\lambda \, \overline{\partial} \zeta_j^\alpha,$$

which is a local 1-form in a neighborhood of M_0 in $\mathcal{U}_j$. The local expression

$$\varphi_j(z,t) = \sum_{\lambda=1}^n \varphi_j^\lambda \otimes \frac{\partial}{\partial z^\lambda}, \tag{1.7.3}$$

defines a global $(0,1)$-vector form on M_0. In fact, the forms φ_j^λ do not depend on the choice of the neighborhood $\mathcal{U}_j$ and ζ^j. Let ζ^k be another set of coordinates on a different $\mathcal{U}_k$, so that $f_{jk}(\zeta^k(z,t), t)$ are the transition functions. We remember that f_{jk} are holomorphic in ζ_k, hence

$$A^{\lambda}_{j\alpha} = \sum_{\beta=1}^{n} A^{\lambda}_{k\beta} \frac{\partial \zeta^{\beta}_{k}}{\partial \zeta^{\alpha}_{j}},$$

which, with a coordinate change, yields

$$\begin{aligned}\varphi^{\lambda}_{j}(z,t) &= \sum_{\alpha,\beta=1}^{n} \frac{\partial \zeta^{\alpha}_{j}}{\partial \zeta^{\beta}_{k}} A^{\lambda}_{j\alpha}\, \overline{\partial} \zeta^{\beta}_{k}(z,t) \\ &= \sum_{\beta=1}^{n} A^{\lambda}_{k\beta}\, \overline{\partial} \zeta^{\beta}_{k}(z,t) = \varphi^{\lambda}_{k}(z,t)\end{aligned}$$

Therefore, we can define $\varphi^{\lambda} = \varphi^{\lambda}_{j}(z,t)$, which is a $(0,1)$-form independent of the coordinates ζ^{j}, and φ has local expression φ_j in a neighborhood $V \times B$, with V an open set in M_0 with coordinates z. It remains to show that the expression of φ does not depend on the local coordinate z on M_0. Let V and W be two open sets in M_0 and z_V, z_W their local coordinates. Under the coordinate change $z_W = z_W(z_V)$, it then turns out that

$$\varphi(t) = \sum_{\beta=1}^{n} \varphi^{\beta}(z_W,t) \otimes \frac{\partial}{\partial z^{\beta}_{W}} = \sum_{\beta=1}^{n} \varphi^{\beta}(z_V,t) \otimes \frac{\partial}{\partial z^{\beta}_{V}},$$

hence $\varphi(t)$ is a global $(0,1)$-vector form on M_0. By its very definition, it holds that

$$i_{\varphi(t)}\, \zeta^{\alpha}_{j}(z,t) = \sum_{\lambda=1}^{n} \varphi^{\lambda} \frac{\partial \zeta^{\alpha}_{j}}{\partial z^{\lambda}} = \overline{\partial} \zeta^{\alpha}_{j}(z,t)$$

or equivalently

$$\left(\overline{\partial} - \sum_{\lambda=1}^{n} \varphi^{\lambda} \otimes \frac{\partial}{\partial z^{\lambda}}\right) \zeta^{\alpha}_{j}(z,t) = 0.$$

It can be proved (see [85, Chapter 4, Proposition 1.2]) that the (local) holomorphic functions on each M_t are defined as the differentiable functions f defined on open sets of M_0 which are solutions to equation

$$\left(\overline{\partial} - \sum_{\lambda=1}^{n} \varphi^{\lambda} \otimes \frac{\partial}{\partial z^{\lambda}}\right) f(z,t) = 0, \tag{1.7.4}$$

i.e., the complex structure on each M_t, for t small enough, is encoded in the $(0,1)$-vector form $\varphi(t)$.

We remark that on the spaces of vector forms on M, i.e., $\mathcal{A}_{*} := \mathcal{A}^{0,*}(T^{1,0}M_0)$, $* \in \{1,\dots,n\}$, a bracket can be defined in the following way. Let $\Psi = \sum \psi^{\alpha}\partial_{\alpha}$ and $\Xi = \sum \xi^{\alpha}\partial_{\alpha}$ be respectively $(0,p)$- and a $(0,q)$-vector forms, where $\partial_{\alpha} = \frac{\partial}{\partial z^{\alpha}}$. Then

$$[\,\Psi\,,\,\Xi\,] := \sum_{\alpha,\beta=1}^{n} \left(\psi^{\alpha} \wedge \partial_{\alpha}\xi^{\beta} - (-1)^{pq}\xi^{\alpha} \wedge \partial_{\alpha}\psi^{\beta}\right)\partial_{\beta} \quad \in \mathcal{A}_{p+q}. \tag{1.7.5}$$

In particular $[\ ,\ ]$ is bilinear and satisfies the following

1. $[\Psi,\Xi] = -(-1)^{pq}[\Xi,\Psi]$,
2. $\overline{\partial}[\Psi,\Xi] = [\overline{\partial}\Psi,\Xi] + (-1)^{p}[\Psi,\overline{\partial}\Xi]$,
3. $(-1)^{pr}[\Psi[\Xi,\Phi]] + (-1)^{qp}[\Xi,[\Phi,\Psi]] + (-1)^{rq}[\Phi,[\Psi,\Xi]]{=}0$,

if $\Psi \in \mathcal{A}_p$, $\Xi \in \mathcal{A}_q$ and $\Phi \in \mathcal{A}_r$.

A classical result (see [85, Chapter 4, Theorem 1.1]) shows that the deformations of the complex structure on a compact complex manifold can be characterized according to the following theorem.

Theorem 1.7.4. *If $(\mathcal{M}, \pi, B)$ is a differentiable family of compact complex manifolds, then the complex structure on each $M_t = \pi^{-1}(t)$ is represented by the vector $(0,1)$-form $\varphi(t) \in \mathcal{A}_1$ just constructed on M_0, such that $\varphi(0) = 0$ and*

$$\overline{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0 \qquad \text{(Maurer-Cartan equation)}. \tag{1.7.6}$$

As for the existence of deformations of compact complex manifolds, we refer to the general theory known as *Kuranishi theory*.

Let M be a compact complex manifold. Fix an Hermitian metric h on M, extend it to $\mathcal{A}_q$ and denote it by the same symbol h. Define and inner product on $\mathcal{A}_q$ by

$$\langle\!\langle \Psi, \Xi \rangle\!\rangle = \int_M h(\Psi, \Xi) * 1,$$

where $\Psi, \Xi \in \mathcal{A}_q$, $*$ is the $\mathbb{C}$-antilinear Hodge operator. We also define the Laplacian on $\mathcal{A}_q$ by

$$\square = \overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^*,$$

where $\overline{\partial}^*$ is the adjoint operator of $\overline{\partial}$ with respect to the Hermitian metric h. The space of harmonic forms is

$$\mathcal{H}^q = \{\Psi \in \mathcal{A}_q : \square\Psi = 0\}.$$

The Hodge theory induces a decomposition on the space $\mathcal{A}_q$ as a direct sum of orthogonal subspaces:

$$\mathcal{A}_q = \mathcal{H}^q \oplus \square\mathcal{A}_q.$$

The operator $G\colon \mathcal{A}_q \to \square\mathcal{A}_q$ is well defined and acts on $\mathcal{A}_q$ as the projection onto $\square\mathcal{A}_q$, whereas the operator H is the well-defined projection operator onto $\mathcal{H}^q$.

Theorem 1.7.5 ([89]). *Let M be a compact complex manifold, $\{\eta_\nu\}$ a base for $\mathcal{H}^1$. Let $\varphi(t)$ be the $(0,1)$-vector form which is a power series solution of the equation*

$$\varphi(t) = \eta(t) + \frac{1}{2}\overline{\partial}^* G[\varphi(t), \varphi(t)], \tag{1.7.7}$$

where $\eta(t) = \sum_{\nu=1}^m t_\nu \eta_\nu$, $|t| < r$, $r > 0$, and let $S = \{t \in B_r(0) : H[\varphi(t), \varphi(t)] = 0\}$. Then for each $t \in S$, $\varphi(t)$ determines a complex structure M_t on M.

The space S is called the *space of Kuranishi*. The proof of Theorem 1.7.5 shows that a $(0,1)$-vector form $\varphi(t)$ satisfying equation (1.7.7) can be constructed as a converging power series

$$\varphi(t) = \sum_{\mu=1}^{\infty} \varphi_\mu(t)$$

in which the forms

$$\varphi_\mu(t) = \sum_{\nu_1 + \cdots + \nu_m = \mu} \varphi_{\nu_1 \dots \nu_m} t_1^{\nu_1} \cdots t_m^{\nu_m}, \quad \varphi_{\nu_1 \dots \nu_m} \in \mathcal{A}_1,$$

are determined via a recursive formula. In fact, if $\{\eta_\nu\}_{\nu=1}^n$ is a basis for $\mathcal{H}^1$ and we set $\psi_1(t) = \sum_{\nu=1}^m t_\nu \eta_\nu$, equation (1.7.7) assures that each term φ_μ can be computed as

$$\varphi_\mu(t) = \frac{1}{2}\overline{\partial}^* G\Big(\sum_{\kappa=1}^{\mu-1} [\varphi_\kappa(t), \varphi_{\mu-\kappa}(t)]\Big). \tag{1.7.8}$$

In general S can have singularities and hence may not have a structure of smooth manifold. Nonetheless, $\{M_t\}_{t\in S}$ can be proven to be a locally complete family of complex manifolds and therefore can be still be interpreted as a complex analytic family, see [89].

Let then $(\mathcal{M},\pi,B)$ be a differentiable family of compact complex manifolds. In order to study the geometry of deformations, it is useful to understand the decompositions of the complexified cotangent bundle $(T_{\mathbb{C}}M)^*$ and how its powers $\wedge^k_{\mathbb{C}}(M)$ vary along with M_t. For simplicity, we suppose that $B = I = (-\epsilon,\epsilon) \subset \mathbb{R}$, for $\epsilon > 0$. In the following, we may refer to one-dimensional differentiable families of complex manifolds $\{M_t\}_{t\in I}$, $I = (-\epsilon,\epsilon)$, $\epsilon > 0$, by the terminology *curves of complex structures*.

Let us denote the central fiber of the family $M_0 = \pi^{-1}(0)$ by M and let us suppose $\varphi(t) \in \mathcal{A}_1$ is the $(0,1)$-vector form describing $(\mathcal{M},\pi,B)$. If we denote by

$$i^k_{\varphi(t)} := \underbrace{i_{\varphi(t)} \circ \cdots \circ i_{\varphi(t)}}_{k \text{ times}}$$

and by $\overline{\varphi(t)} \in \mathcal{A}^{1,0}(T^{0,1}M)$ the conjugate of $\varphi(t)$, in the following operators

$$e^{i_{\varphi(t)}} = \sum_{k=0}^{\infty} \frac{1}{k!} i^k_{\varphi(t)} \qquad \text{and} \qquad e^{i_{\overline{\varphi(t)}}} = \sum_{k=0}^{\infty} \frac{1}{k!} i^k_{\overline{\varphi(t)}}$$

the summations are finite, since the dimension of M is finite. As in [122, Definition 2.8], we recall the *extension map*

$$e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}} \colon \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q}(M_t), \tag{1.7.9}$$

where, if $\alpha = \alpha_{i_1\dots i_p j_1\dots j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$ is the local expression of α a (p,q)-differential form on M, we set

$$e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\alpha) = \alpha_{i_1\dots i_p j_1\dots j_q} e^{i_{\varphi(t)}}(dz^{i_1} \wedge \cdots \wedge dz^{i_p}) \wedge e^{i_{\overline{\varphi(t)}}}(d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}). \tag{1.7.10}$$

Note that the local definition of $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\alpha)$ gives rise to a global (p,q)-form on M_t, since $\varphi(t)$ is a global $(0,1)$-vector form.

The following lemma relates (p,q)-forms on the central fiber M and (p,q)-forms on any alement M_t of the family $\mathcal{M}$, see [122, Lemma 2.9, 2.10].

Lemma 1.7.6. *For any p,q and for t small, the map $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}} \colon \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q}(M_t)$ is a real linear isomorphism.*

Moreover, the following decompositions hold

$$\mathcal{A}^k_{\mathbb{C}}(M) = \oplus_{p+q=k} \mathcal{A}^{p,q}(M_t), \qquad k \in \{1,\dots,n\}. \tag{1.7.11}$$

Remark 1.7.7. We observe that, for a $(0,1)$-vector form $\varphi(t) \in \mathcal{A}_1$ on M_0 such that $\varphi(0) = 0$, the Maurer-Cartan equation (1.7.6) is equivalent to the integrability of the complex structure J_t on M_t, i.e.,

$$(d\alpha)^{0,2} = 0 \qquad \forall \alpha \in \mathcal{A}^{1,0}(M_t). \tag{1.7.12}$$

Indeed, from Lemma 1.7.6 it immediately follows $(I - \varphi)\lrcorner : \Gamma(T^{1,0}M) \to \Gamma(T^{1,0}M_t)$ is an isomorphism for t small, and for $X, Y \in \Gamma(T^{1,0}M)$

$$-d(\alpha + e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\alpha))(X - \varphi(t)(X), Y - \varphi(t)(Y)) = \alpha\Big((\bar\partial\varphi(t) - \frac{1}{2}[\varphi(t),\varphi(t)])(X,Y)\Big).$$

See also [75, Proposition 6.1.2]. Furthermore, for a $(0,1)$-vector form satisfying (1.7.7), the defining property of S, i.e., $H[\varphi(t),\varphi(t)] = 0$, is equivalent to the integrability condition given by the Maurer-Cartan equation (1.7.6) (see [85, Chapter 4, Proposition 2.5]).

Chapter 2

Deformations of special Hermitian metrics

Small deformations of the complex structure do not always preserve special metric properties in the Hermitian non-Kähler setting. In particular, the existence of SKT metrics, astheno-Kähler metrics, and balanced metrics, on complex manifolds has been shown to be unstable under small deformations, see [8, 54]) (note that sufficient stability conditions have been proved for balanced metrics [20, 127, 121, 58, 111]. In this chapter, for each class of such special metrics, we prove necessary conditions for the existence of smooth curves of SKT metrics $\{\omega_t\}_t$ (respectively astheno-Kähler metrics, or balanced metrics) which start with a fixed SKT metric ω (respectively, astheno-Kähler metric, or balanced metric) for $t = 0$, along a differentiable family of compact complex manifolds $\{M_t\}_t$, see Theorems 2.2.1, 2.3.1, and 2.4.1. From such theorems, as an immediate consequence, we obtain the obstructions gathered in Corollaries 2.2.2, 2.3.2, and 2.4.2, thus relating the stability under deformations of the property of admitting special metrics and the Dolbeault an Bott-Chern cohomologies of the starting manifold. As an application, we provide examples of obstructions on several concrete examples: for SKT and astheno-Kähler metrics, we focus on families of nilmanifolds with invariant complex structure of complex dimension 4, whereas for balanced metrics, we characterize the obstructions to the existence of curves of balanced metrics on the complex parallelixable solvmanifolds.

2.1 Formulas for ∂_t and $\overline{\partial}_t$ operators on curves of deformations

Let $(\mathcal{M}, \pi, I)$ be a differentiable family of compact complex manifolds parametrized by $\varphi(t)$, for $t \in I$, $I := (-\epsilon, \epsilon)$, $\epsilon > 0$, as seen in section 1.7. We need to recall formulas for the differential operators ∂_t and $\overline{\partial}_t$, defined as

$$\begin{aligned}\partial_t &:= \pi_t^{p+1,q} \circ d \colon \mathcal{A}^{p,q}(M_t) \to \mathcal{A}^{p+1,q}(M_t),\\ \overline{\partial}_t &:= \pi_t^{p,q+1} \circ d \colon \mathcal{A}^{p,q}(M_t) \to \mathcal{A}^{p,q+1}(M_t),\end{aligned}$$

for any p, q, with $\pi_t^{p+1,q}$ and $\pi_t^{p,q+1}$ the usual projections of $d(\mathcal{A}^{p,q}(M_t))$ with respect to the decompositions in (p,q) forms on M_t (1.7.11).

We take as main reference [122]. Starting from $(0,0)$-differential forms, i.e., differentiable complex functions, we have

$$\begin{aligned}\partial_t f &= e^{i_\varphi}\Big((I - \varphi\overline{\varphi})^{-1} \lrcorner (\partial - \overline{\varphi} \lrcorner \overline{\partial}) f\Big),\\ \overline{\partial}_t f &= e^{i_{\overline{\varphi}}}\Big((I - \overline{\varphi}\varphi)^{-1} \lrcorner (\overline{\partial} - \varphi \lrcorner \partial) f\Big),\end{aligned}$$

where $\varphi\overline{\varphi} = \overline{\varphi} \lrcorner \varphi$, $\overline{\varphi}\varphi = \varphi \lrcorner \overline{\varphi}$ and we omit the dependence on t of φ, see [122, Equation (2.13)]. We will denote by $\rightthreetimes$ the simultaneous contraction on each component of a complex differential form, i.e.

$$\varphi \rightthreetimes \alpha := \alpha_{i_1 \dots i_p j_1 \dots j_q} \varphi \lrcorner dz^{i_1} \wedge \dots \wedge \varphi \lrcorner dz^{i_p} \wedge \overline{\varphi} \lrcorner \overline{z}^{j_1} \wedge \dots \wedge \overline{\varphi} \lrcorner d\overline{z}^{j_q},$$

for any (p,q)-differential form locally written as $\alpha = \alpha_{i_1 \dots i_p j_1 \dots j_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \dots \wedge d\overline{z}^{j_q}$. This contraction is well-defined and it can be used to describe the extension map, in fact

$$e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}} = (I + \varphi + \overline{\varphi}) \rightthreetimes. \tag{2.1.1}$$

With these notations, from the proof of [122, Proposition 2.13], we can summarize the action of the operators ∂_t and $\overline{\partial}_t$ on differential forms $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}\alpha \in \mathcal{A}^{p,q}(M_t)$, with $\alpha \in \mathcal{A}^{p,q}(M)$, follows

$$\partial_t(e^{i_\varphi|i_{\overline{\varphi}}}\alpha) = e^{i_\varphi|i_{\overline{\varphi}}}\Big((I - \varphi\overline{\varphi})^{-1} \rightthreetimes([\overline{\partial}, i_{\overline{\varphi}}] + \partial)(I - \varphi\overline{\varphi}) \rightthreetimes \alpha\Big), \tag{2.1.2}$$

$$\overline{\partial}_t(e^{i_\varphi|i_{\overline{\varphi}}}\alpha) = e^{i_\varphi|i_{\overline{\varphi}}}\Big((I - \overline{\varphi}\varphi)^{-1} \rightthreetimes([\partial, i_{\varphi}] + \overline{\partial})(I - \overline{\varphi}\varphi) \rightthreetimes \alpha\Big). \tag{2.1.3}$$

2.2 Deformations of strong Kähler with torsion metrics

Let us fix (M, J, g, ω) a compact Hermitian manifold and suppose that g is SKT, i.e. $\partial\overline{\partial}\omega = 0$. We want to find necessary conditions under which the property of being SKT is stable for a smooth family of Hermitian metrics $\{\omega_t\}_{t \in I}$ such that $\omega_0 = \omega$, along a deformation of the complex structure parametrized by a $(0,1)$-vector form $\varphi(t)$. Suppose that each ω_t is SKT for any $t \in I$, i.e. $\partial_t\overline{\partial}_t\omega_t = 0$. Using the Taylor series expansion and differentiating this expression with respect to t, we obtain the following.

Theorem 2.2.1 ([118])**.** *Let (M, J, g, ω) be a compact Hermitian manifold with g a SKT metric. Let $\{M_t\}_{t \in I}$ be a differentiable family of compact complex manifolds parametrized by $\varphi(t) \in \mathcal{A}_1$, for $t \in I = (-\epsilon, \epsilon)$, $\epsilon > 0$. Let $\{\omega_t\}_{t \in I}$ be a smooth family of Hermitian metrics on each M_t written as*

$$\omega_t = e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}\,(\omega(t)),$$

where $\omega(t)$ has local expression $\omega_{ij}(t)\, dz^i \wedge d\overline{z}^j \in \mathcal{A}^{1,1}(M)$. Denote by $\omega'(t) := \frac{\partial}{\partial t}\omega_{ij}(t)\, dz^i \wedge d\overline{z}^j \in \mathcal{A}^{1,1}(M)$. Then, if the metrics ω_t are SKT for every $t \in I$, the following condition must hold

$$2i\mathfrak{Im}(\partial \circ i_{\varphi'(0)} \circ \partial)(\omega) = \partial\overline{\partial}\omega'(0). \tag{2.2.1}$$

As a consequence, we have the following cohomological obstruction.

Corollary 2.2.2 ([118])**.** *Let (M, J, g, ω) be a compact Hermitian manifold. A necessary condition for the existence of a smooth family of SKT metrics which equals ω in $t = 0$ along the family of deformations $t \mapsto \varphi(t)$ is that the following equation must hold*

$$\big[\mathfrak{Im}(\partial \circ i_{\varphi'(0)} \circ \partial)(\omega)\big]_{H^{2,2}_{BC}(M)} = 0.$$

Proof of Theorem 2.2.1. The metrics ω_t are SKT for every $t \in I$, i.e., $\partial_t\overline{\partial}_t\omega_t = 0$. This implies

$$\frac{\partial}{\partial t}(\partial_t\overline{\partial}_t\omega_t)_{|t=0} = 0. \tag{2.2.2}$$

Let us compute equation (2.2.2) using the expressions (2.1.2) and (2.1.3) for ∂_t and $\overline{\partial}_t$. First we calculate $\overline{\partial}_t(\omega_t)$

$$\overline{\partial}_t(\omega_t) = e^{i_\varphi|i_{\overline{\varphi}}}\Big((I - \overline{\varphi}\varphi)^{-1} \rightthreetimes([\partial, i_\varphi] + \overline{\partial})(I - \overline{\varphi}\varphi) \rightthreetimes \omega(t)\Big),$$

and then $\partial_t\overline{\partial}_t(\omega_t)$,

$$\partial_t\overline{\partial}_t(\omega_t) = e^{i_\varphi|i_{\overline{\varphi}}}\Big((I-\varphi\overline{\varphi})^{-1}\lrcorner([\overline{\partial}, i_{\overline{\varphi}}]+\partial)(I-\varphi\overline{\varphi})\lrcorner(I-\overline{\varphi}\varphi)^{-1}\lrcorner([\partial, i_\varphi]+\overline{\partial})(I-\overline{\varphi}\varphi)\lrcorner\omega(t)\Big).$$

Now, to compute equation (2.2.2), we develop $\partial_t\overline{\partial}_t(\omega_t)$ in Taylor series centered in $t=0$ up to the first order. Note that

$$\varphi(t) = t\varphi'(0) + o(t)$$

implies

$$(I-\varphi\overline{\varphi}) = (I-\overline{\varphi}\varphi) = (I-\varphi\overline{\varphi})^{-1} = (I-\overline{\varphi}\varphi)^{-1} = I + o(t). \tag{2.2.3}$$

Therefore we get

$$\begin{aligned}
\partial_t\overline{\partial}_t(\omega_t) &= (I + t\varphi'(0) + t\overline{\varphi'(0)})\lrcorner([\overline{\partial}, t\overline{\varphi'(0)}\lrcorner] + \partial)([\partial, t\varphi'(0)\lrcorner] + \overline{\partial})\Big(\omega(0) + t\omega'(0)\Big) + o(t)\\
&= (I + t\varphi'(0) + t\overline{\varphi'(0)})\lrcorner([\overline{\partial}, t\overline{\varphi'(0)}\lrcorner] + \partial)\Big([\partial, t\varphi'(0)\lrcorner]\omega(0) + \overline{\partial}\omega(0) + t\overline{\partial}\omega'(0)\Big) + o(t)\\
&= (I + t\varphi'(0) + t\overline{\varphi'(0)})\lrcorner\Big(-t\partial(\varphi'(0)\lrcorner\partial\omega(0)) + t\overline{\partial}(\overline{\varphi'(0)}\lrcorner\overline{\partial}\omega(0)) + t\partial\overline{\partial}\omega'(0)\Big) + o(t)\\
&= -t\partial(\varphi'(0)\lrcorner\partial\omega(0)) + t\overline{\partial}(\overline{\varphi'(0)}\lrcorner\overline{\partial}\omega(0)) + t\partial\overline{\partial}\omega'(0) + o(t),
\end{aligned}$$

implying

$$0 = \frac{\partial}{\partial t}(\partial_t\overline{\partial}_t\omega_t)_{|t=0} = -\partial(\varphi'(0)\lrcorner\partial\omega(0)) + \overline{\partial}(\overline{\varphi'(0)}\lrcorner\overline{\partial}\omega(0)) + \partial\overline{\partial}\omega'(0),$$

which is equivalent to equation (2.2.1). □

We now apply Corollary 2.2.2 and Theorem 2.2.1 to study two 4-dimensional complex nilmanifolds admitting invariant SKT metrics. In particular, we study obstructions along a specific family of deformations on a family of nilmanifolds introduced in [55, Section 2.3] and on a quotient of the product of two copies of the real Heisenberg group $\mathbb{H}(3;\mathbb{R})$ and $\mathbb{R}^2$ presented in [125, Example 8].

2.2.1 Example 1

Let us consider the Lie algebra $\mathfrak{g}$ endowed with integrable almost complex structure J such that $\mathfrak{g}^*$ is spanned by $\{\eta^1,\dots,\eta^4\}$, a set of $(1,0)$ complex differential forms with structure equations

$$\begin{cases}
d\eta^i &= 0, \qquad i\in\{1,2,3\},\\
d\eta^4 &= a_1\eta^{12} + a_2\eta^{13} + a_3\eta^{1\overline{1}} + a_4\eta^{1\overline{2}} + a_5\eta^{1\overline{3}}\\
&\quad + a_6\eta^{23} + a_7\eta^{2\overline{1}} + a_8\eta^{2\overline{2}} + a_9\eta^{2\overline{3}}\\
&\quad + a_{10}\eta^{3\overline{1}} + a_{11}\eta^{3\overline{2}} + a_{12}\eta^{3\overline{3}},
\end{cases} \tag{2.2.4}$$

with $a_i\in\mathbb{C}$ for $i\in\{1,\dots,12\}$. In particular, $\mathfrak{g}$ is a 2-step nilpotent Lie algebra depending on the complex parameters $a_1,\dots,a_{12}$. If we denote by G the simply-connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, then for any $a_1,\dots,a_{12}\in\mathbb{Q}[i]$, by Malcev's theorem [98, Theorem 7], there exists a uniform discrete subgroup Γ of G such that $M=\Gamma\backslash G$ is nilmanifold. As in [55, Theorem 2.7], the left-invariant Hermitian metric on M

$$g = \frac{1}{2}\sum_{j=1}^{4}(\eta^j\otimes\overline{\eta}^j + \overline{\eta}^j\otimes\eta^j)$$

is Astheno Kähler, i.e., the fundamental form of g

$$\omega = \frac{i}{2}\sum_{j=1}^{4}\eta^j\wedge\overline{\eta}^j \tag{2.2.5}$$

is such that $\partial\overline{\partial}\omega^2 = 0$, if and only if the following equation holds

$$|a_1|^2 + |a_2|^2 + |a_4|^2 + |a_5|^2 + |a_6|^2 + |a_7|^2 + |a_9|^2 + |a_{10}|^2 + |a_{11}|^2 = 2\mathfrak{Re}(a_3\overline{a}_8 + a_3\overline{a}_{12} + a_8\overline{a}_{12}). \quad (2.2.6)$$

Moreover, if $a_8 = 0$, the Astheno-Kähler metric g is SKT if and only if

$$a_1 = a_4 = a_6 = a_7 = a_9 = a_{11} = 0.$$

Hence, if $a_i = 0$ for $i \in \{1,4,6,7,8,9,11\}$ and

$$|a_2|^2 + |a_5|^2 + |a_{10}|^2 = 2\mathfrak{Re}(a_3\overline{a}_{12}), \quad (2.2.7)$$

from equation (2.2.6), the metric g is SKT, i.e., $\partial\overline{\partial}\omega = 0$. From now on, we will consider the nilmanifold (M,J), with Hermitian SKT metric ω.

The structure equations (2.2.4) boil down to

$$\begin{cases} d\eta^i &= 0, \qquad i \in \{1,2,3\}, \\ d\eta^4 &= a_2\eta^{13} + a_3\eta^{1\overline{1}} + a_5\eta^{1\overline{3}} + a_{10}\eta^{3\overline{1}} + a_{12}\eta^{3\overline{3}}. \end{cases} \quad (2.2.8)$$

We consider now the following invariant $(0,1)$-vector form given by

$$\varphi(r,s) = r\overline{\eta}^1 \otimes Z_1 + s\overline{\eta}^3 \otimes Z_3, \qquad (r,s) \in \mathbb{C}^2,\ |r| < 1,\ |s| < 1,$$

where Z_j is the dual of η^j in $\mathfrak{g}$, for $j \in \{1,2,3,4\}$. By Lemma 1.7.6 the invariant forms $\eta^j_{r,s} := \eta^j + i_\varphi(\eta^j)$, for $j \in \{1,2,3,4\}$

$$\begin{cases} \eta^1_{r,s} &= \eta^1 + r\overline{\eta}^1, \\ \eta^2_{r,s} &= \eta^2, \\ \eta^3_{r,s} &= \eta^3 + s\overline{\eta}^3, \\ \eta^4_{r,s} &= \eta^4, \end{cases}$$

form a coframe of $(T^{1,0}M_t)^*$. It is clear that

$$\begin{cases} \eta^1 &= \frac{1}{1-|r|^2}(\eta^1_{r,s} - r\overline{\eta}^1_{r,s}), \\ \eta^2 &= \eta^2_{r,s}, \\ \eta^3 &= \frac{1}{1-|s|^2}(\eta^3_{r,s} - s\overline{\eta}^3_{r,s}), \\ \eta^4 &= \eta^4_{r,s}. \end{cases}$$

Therefore, it can be easily seen that the structure equations for the coframe $\{\eta^1_{r,s}, \eta^2_{r,s}, \eta^3_{r,s}, \eta^4_{r,s}\}$ are

$$\begin{cases} d\eta^i_{r,s} &= 0, \qquad i \in \{1,2,3\}, \\ d\eta^4_{r,s} &= \frac{a_2+\overline{r}a_{10}-\overline{s}a_5}{(1-|r|^2)(1-|s|^2)}\,\eta^{13}_{r,s} + \frac{a_3}{1-|r|^2}\,\eta^{1\overline{1}}_{r,s} + \frac{a_5-sa_2-\overline{r}sa_{10}}{(1-|r|^2)(1-|s|^2)}\,\eta^{1\overline{3}}_{r,s} + \\ &\quad + \frac{a_{10}+ra_2-r\overline{s}a_5}{(1-|r|^2)(1-|s|^2)}\,\eta^{3\overline{1}}_{r,s} + \frac{a_{12}}{1-|s|^2}\,\eta^{3\overline{3}}_{r,s} + \frac{-ra_5+sa_{10}+rsa_2}{(1-|r|^2)(1-|s|^2)}\,\eta^{\overline{1}\overline{3}}_{r,s}. \end{cases}$$

For the integrability condition $(d\eta^i_{r,s})^{0,2} = 0$, which is equivalent to check the Maurer Cartan equation for φ by Remark 1.7.7, we must have that

$$-ra_5 + sa_{10} + rsa_2 = 0. \quad (2.2.9)$$

We begin studying this equation by noticing that, if we set $F(r,s) = -ra_5 + sa_{10} + rsa_2$, the gradient ∇F in $(r,s) = (0,0)$ is

$$\begin{pmatrix} F_r(0,0) \\ F_s(0,0) \end{pmatrix} = \begin{pmatrix} -a_5 \\ a_{10} \end{pmatrix}.$$

We distinguish two cases, depending on whether $\nabla F(0,0) = 0$ or $\nabla F(0,0) \neq 0$. We observe that in the first case, the solution set, which we will denote by B, might not be a smooth manifold, whereas it happens in the latter case.

Case (i)

$\nabla F(0,0) = 0$, i.e., $a_5 = a_{10} = 0$. The solutions of (2.2.9) are

$$S = \{(r,s) \in \mathbb{C}^2 : rsa_2 = 0, |r|, |s| < \delta\},$$

for $\delta > 0$ sufficiently small. The corresponding $(0,1)$-vector form which parametrizes the deformation is $\varphi = r\overline{\eta}^1 \otimes Z_1 + s\overline{\eta}^3 \otimes Z_3$. If we consider the segment $\gamma\colon (-\epsilon, \epsilon) \to S$, $\gamma(t) = (tu, tv)$ for $(u,v) \in S$, we define the curve of deformations

$$t \mapsto \varphi(t) = tu\,\overline{\eta}^1 \otimes Z_1 + tv\,\overline{\eta}^3 \otimes Z_3.$$

In this case, $\varphi'(0) = u\,\overline{\eta}^1 \otimes Z_1 + v\,\overline{\eta}^3 \otimes Z_3$. With structure equation

$$\begin{cases} d\eta^i &= 0, \quad i \in \{1,2,3\}, \\ d\eta^4 &= a_2\eta^{13} + a_3\eta^{1\overline{1}} + a_{12}\eta^{3\overline{3}}, \end{cases}$$

we compute $\partial \circ i_{\varphi'(0)} \circ \partial(\omega)$. It turns out that this term vanishes, therefore Corollary 2.2.2 gives no obstructions to the existence of curve of SKT metrics along the curve of deformations $t \mapsto \varphi(t)$.

Case (ii)

$\nabla F(0,0) \neq 0$, i.e., $(a_5, a_{10}) \neq (0,0)$.

We begin by studying the case $a_5 \neq 0$. The set

$$S = \left\{(r,s) \in \mathbb{C}^2 : r = \frac{sa_{10}}{a_5 - sa_2},\ |r| < \delta, |s| < \delta'\right\},$$

for $\delta, \delta' > 0$ sufficiently small, is the set of the solutions of equation (2.2.9)

$$-ra_5 + sa_{10} + rsa_2 = 0.$$

If we consider the smooth curve $\gamma\colon (-\epsilon, \epsilon) \to S$,

$$\gamma(t) = (\frac{tua_{10}}{a_5 - tua_2}, tu) \tag{2.2.10}$$

with $u \in \mathbb{C}$, we have that

$$t \mapsto \varphi(t) = \frac{tua_{10}}{a_5 - tua_2}\,\overline{\eta}^1 \otimes Z_1 + tu\overline{\eta}^3 \otimes Z_3$$

is a smooth curve of deformations with $\varphi'(0) = \frac{ua_{10}}{a_5}\,\overline{\eta}^1 \otimes Z_1 + u\,\overline{\eta}^3 \otimes Z_3$. By the usual computations and structure equations

$$\begin{cases} d\eta^i &= 0, \quad i \in \{1,2,3\}, \\ d\eta^4 &= a_2\eta^{13} + a_3\eta^{1\overline{1}} + a_5\eta^{1\overline{3}} + a_{10}\eta^{3\overline{1}} + a_{12}\eta^{3\overline{3}}, \end{cases}$$

we obtain that

$$\partial \circ i_{\varphi'(0)} \circ \partial(\omega) = iua_2\frac{|a_{10}|^2 - |a_5|^2}{a_5}\eta^{13\overline{1}\overline{3}}.$$

We observe that the real form $\eta^{13\overline{1}\overline{3}}$ is closed with respect to ∂ and $\overline{\partial}$. Moreover,

$$(\partial\overline{\partial}*)\eta^{13\overline{1}\overline{3}} = (|a_2|^2 + |a_3|^2 + |a_{10}|^2 - 2\Re(a_3\overline{a}_{12}))\eta^{123\overline{1}\overline{2}\overline{3}} = 0,$$

by equation (2.2.7). Therefore $\eta^{13\bar{1}\bar{3}}$ is harmonic with respect to the Bott-Chern Laplacian and, via the canonical isomorphism, the class $[\eta^{13\bar{1}\bar{3}}]_{BC}$ is a non-vanishing class in $H^{2,2}_{BC}(M)$. Hence, if

$$\mathfrak{Im}\left(iua_2\frac{|a_{10}|^2-|a_5|^2}{a_5}\right)\neq 0,$$

by Corollary (2.2.2) there exist no family of SKT metrics ω_t along $t\mapsto\varphi(t)$ such that $\omega_0=\omega$.

If instead we assume that $a_{10}\neq 0$, we have that equation (2.2.9)

$$-ra_5+sa_{10}+rsa_2=0$$

admits solutions

$$S=\left\{(r,s)\in\mathbb{C}^2 : s=\frac{ra_5}{a_{10}+ra_2},\ |r|<\delta,|s|<\delta'\right\},$$

with $\delta,\delta'>0$ sufficiently small.

If $\gamma\colon(-\epsilon,\epsilon)\to S$ is the smooth curve $\gamma(t)=(tv,\frac{tva_5}{a_{10}+tva_2})$ with $v\in\mathbb{C}$, we define the curve of deformations by

$$t\mapsto\varphi(t)=tv\bar{\eta}^1\otimes Z_1+\frac{tva_5}{a_{10}+tva_2}\bar{\eta}^3\otimes Z_3. \tag{2.2.11}$$

We notice that $\varphi'(0)=v\bar{\eta}^1\otimes Z_1+v\frac{a_5}{a_{10}}\bar{\eta}^3\otimes Z_3$. With the aid of structure equations (2.2.8), we can check that

$$\partial\circ i_{\varphi'(0)}\circ\bar{\partial}(\omega)=iva_2\frac{|a_{10}|^2-|a_5|^2}{a_{10}}\eta^{13\bar{1}\bar{3}}.$$

Since $\eta^{13\bar{1}\bar{3}}\in\mathcal{H}^{2,2}_{BC}(M,g)$ and $[\eta^{13\bar{1}\bar{3}}]_{BC}$ does not represent the class $0\in H^{2,2}_{BC}$, therefore, if

$$\mathfrak{Im}\left(iva_2\frac{|a_{10}|^2-|a_5|^2}{a_{10}}\right)\neq 0,$$

by Corollary 2.2.2, there is no curve of SKT metrics ω_t along the curve of deformations $t\mapsto\varphi(t)$ such that $\omega_0=\omega$.

Summing up, we gather what we obtained.

Theorem 2.2.3 ([118]). *Let (M,J) be an element of the family of nilmanifolds with structure equations*

$$\begin{cases} d\eta^i &= 0, \qquad i\in\{1,2,3\},\\ d\eta^4 &= a_2\eta^{13}+a_3\eta^{1\bar{1}}+a_5\eta^{1\bar{3}}+a_{10}\eta^{3\bar{1}}+a_{12}\eta^{3\bar{3}},\end{cases}$$

$a_2,a_3,a_5,a_{10},a_{12}\in\mathbb{Q}[i]$ such that $|a_2|^2+|a_5|^2+|a_{10}|^2=2\mathfrak{Re}(a_3\bar{a}_{12})$, endowed with the Hermitian SKT metric $\omega=\frac{i}{2}\sum_{j=1}^4\eta^{j\bar{j}}$. Then

- *if $a_5\neq 0$ and $u\in\mathbb{C}$, there exist no smooth curve of SKT metrics ω_t such that $\omega_0=\omega$ along the curve of deformation $t\mapsto\varphi(t)=\frac{tua_{10}}{a_5-tua_2}\bar{\eta}^1\otimes Z_1+tu\bar{\eta}^3\otimes Z_3$ for $t\in(-\epsilon,\epsilon),\epsilon>0$, if*

$$\mathfrak{Im}\left(iua_2\frac{|a_{10}|^2-|a_5|^2}{a_5}\right)\neq 0;$$

- *if $a_{10}\neq 0$ and $v\in\mathbb{C}$, there exist no smooth curve of SKT metrics ω_t such that $\omega_0=\omega$ along the curve of deformation $t\mapsto\varphi(t)=tv\bar{\eta}^1\otimes Z_1+\frac{tva_5}{a_{10}+tva_2}\bar{\eta}^3\otimes Z_3$ for $t\in(-\epsilon,\epsilon),\epsilon>0$, if*

$$\mathfrak{Im}\left(iva_2\frac{|a_{10}|^2-|a_5|^2}{a_{10}}\right)\neq 0.$$

2.2.2 Example 2

Let us consider the group $G := \mathbb{H}(3;\mathbb{R}) \times \mathbb{H}(3;\mathbb{R}) \times \mathbb{R}^2$, where $\mathbb{H}(3;\mathbb{R})$ is the 3-dimensional real Heisenberg group. We fix a basis $\{e^1, \dots, e^8\}$ for $\mathfrak{g}^*$, the dual of the Lie algebra $\mathfrak{g}$ of G such that

$$\begin{cases} de^1 = de^2 = de^3 = de^4 = de^5 = de^7 = 0, \\ de^6 = -\frac{1}{2}e^{12}, \quad de^8 = -\frac{1}{2}e^{34}. \end{cases}$$

Due to [98, Theorem 7], there exists a lattice Γ of G such that the quotient $M = \Gamma \backslash G$ is a compact manifold. In particular, M is a real 8-dimensional nilmanifold.

If we make use of the standard real coordinates $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$ on the two copies of $\mathbb{H}(3;\mathbb{R})$ and $\{x_7, x_8\}$ on $\mathbb{R}^2$, the coframe $\{e^1, \dots, e^8\}$ can be written as

$$\begin{cases} e^1 = dx^1, & e^2 = dx^2, & e^6 = dx^3 - x^1 dx^2, \\ e^3 = dx^4, & e^4 = dx^5, & e^8 = dx^6 - x^4 dx^5, \\ e^5 = dx^7, & e^7 = dx^8. \end{cases}$$

Notice that it defines a global left-invariant coframe of differential 1-forms on G, and therefore on M.

Let us define an almost complex structure J on $\mathfrak{g}^*$ by setting the following basis for $(\mathfrak{g}^{1,0})^*$

$$\begin{cases} \eta^1 := e^1 + ie^2, & \eta^2 := e^3 + ie^4, \\ \eta^3 := e^5 + ie^6, & \eta^4 := e^7 + ie^8. \end{cases}$$

Let Z_j be the dual of η^j in $\mathfrak{g}$, for $j \in \{1,2,3,4\}$. This position gives rise to a left-invariant integrable almost complex structure on G, hence it descends to the quotient M. With an abuse of notation we will denote the latter by J.

We find that the holomorphic coordinates on M which induce J are

$$\begin{cases} z^1 = x^1 + ix^2, \\ z^2 = x^4 + ix^5, \\ z^3 = x^7 + \frac{1}{2}(x^2)^2 + i(x^3 - x^1x^2), \\ z^4 = x^8 + \frac{1}{2}(x^5)^2 + i(x^6 - x^4x^5). \end{cases} \tag{2.2.12}$$

We point out that the structure equations for (M, J) are

$$\begin{cases} d\eta^1 = d\eta^2 = 0, \\ d\eta^3 = -\frac{1}{2}\eta^{1\bar{1}}, \\ d\eta^3 = -\frac{1}{2}\eta^{2\bar{2}}. \end{cases} \tag{2.2.13}$$

Let us now consider a generic Hermitian invariant metric g with associated fundamental form

$$\omega = \frac{i}{2}\sum_{j=1}^{4} a_{j\bar{j}}\, \eta^{j\bar{j}} + \frac{1}{2}\sum_{j<k}\left(\alpha_{j\bar{k}}\, \eta^{j\bar{k}} - \overline{\alpha}_{j\bar{k}}\, \eta^{k\bar{j}}\right),$$

whose coefficients $\alpha_{i\bar{j}}$ are such that the matrix representing g

$$\begin{pmatrix} \alpha_{1\bar{1}} & -i\alpha_{1\bar{2}} & -i\alpha_{1\bar{3}} & -i\alpha_{1\bar{4}} \\ i\overline{\alpha}_{1\bar{2}} & \alpha_{2\bar{2}} & -i\alpha_{2\bar{3}} & -i\alpha_{2\bar{4}} \\ i\overline{\alpha}_{1\bar{3}} & i\overline{\alpha}_{2\bar{3}} & \alpha_{3\bar{3}} & -i\alpha_{3\bar{4}} \\ i\overline{\alpha}_{1\bar{4}} & i\overline{\alpha}_{2\bar{4}} & i\overline{\alpha}_{3\bar{4}} & \alpha_{4\bar{4}} \end{pmatrix}$$

is positive definite.

It is straightforward to check with the aid of (2.2.13), that g is a SKT metric if and only if

$$\mathfrak{Im}(\alpha_{3\bar{4}}) = 0.$$

We construct a $(0,1)$-vector form

$$\begin{aligned}\varphi(\mathbf{t}) &= t_{11}\overline{\eta}^1 \otimes Z_1 + t_{22}\overline{\eta}^2 \otimes Z_2 + t_{32}\overline{\eta}^2 \otimes Z_3 + t_{33}\overline{\eta}^3 \otimes Z_3 \\ &+ t_{34}\overline{\eta}^4 \otimes Z_3 + t_{41}\overline{\eta}^1 \otimes Z_4 + t_{43}\overline{\eta}^3 \otimes Z_4 + t_{44}\overline{\eta}^4 \otimes Z_4,\end{aligned}$$

for $\mathbf{t} = (t_{11}, t_{22}, t_{32}, t_{33}, t_{34}, t_{41}, t_{43}, t_{44})$ in sufficiently small ball B centered in $0 \in \mathbb{C}^8$. Using the holomorphic coordinates (2.2.12), it is a computation to show that φ satisfies Maurer-Cartan equation. As a side note, thanks to [42, Theorem 1.1], we point out $\varphi(\mathbf{t})$ parametrizes a locally complete family of complex analytic deformations. We construct the segment $\gamma\colon(-\epsilon,\epsilon) \to B$, where

$$t \mapsto \gamma(t) = t(a_{11}, a_{22}, a_{32}, a_{33}, a_{34}, a_{41}, a_{43}, a_{44}),$$

with $(a_{11}, a_{22}, a_{32}, a_{33}, a_{34}, a_{41}, a_{43}, a_{44}) \in \mathbb{C}^8$. The corresponding curve of deformations is

$$\begin{aligned}t \mapsto \varphi(t) &= t(a_{11}\overline{\eta}^1 \otimes Z_1 + a_{22}\overline{\eta}^2 \otimes Z_2 + a_{32}\overline{\eta}^2 \otimes Z_3 + a_{33}\overline{\eta}^3 \otimes Z_3 \\ &+ a_{34}\overline{\eta}^4 \otimes Z_3 + a_{41}\overline{\eta}^1 \otimes Z_4 + a_{43}\overline{\eta}^3 \otimes Z_4 + a_{44}\overline{\eta}^4 \otimes Z_4)\end{aligned}$$

whose derivative in $t = 0$ is clearly

$$\begin{aligned}\varphi'(0) &= a_{11}\overline{\eta}^1 \otimes Z_1 + a_{22}\overline{\eta}^2 \otimes Z_2 + a_{32}\overline{\eta}^2 \otimes Z_3 + a_{33}\overline{\eta}^3 \otimes Z_3 \\ &+ a_{34}\overline{\eta}^4 \otimes Z_3 + a_{41}\overline{\eta}^1 \otimes Z_4 + a_{43}\overline{\eta}^3 \otimes Z_4 + a_{44}\overline{\eta}^4 \otimes Z_4.\end{aligned}$$

Via structure equations (2.2.13) and the expression of $\varphi'(0)$, we obtain that

$$\begin{aligned}&2i\mathfrak{Im}((\partial \circ i_{\varphi'(0)} \circ \partial)(\omega)) = \\ &\frac{1}{8}(i\alpha_{3\bar{3}}(a_{34} + \overline{a}_{34}) + i\alpha_{4\bar{4}}(a_{43} + \overline{a}_{43}) + \alpha_{3\bar{4}}(a_{33} + \overline{a}_{44}) - \overline{\alpha}_{3\bar{4}}(a_{44} + \overline{a}_{33}))\, \eta^{12\bar{1}\bar{2}}.\end{aligned} \tag{2.2.14}$$

We observe that $\eta^{12\bar{1}\bar{2}} = \frac{1}{4}\partial\overline{\partial}(\eta^{3\bar{4}})$, therefore the real $(2,2)$-form $\eta^{12\bar{1}\bar{2}}$ represents the vanishing class in $H^{2,2}_{BC}(M)$. Hence, Corollary 2.2.2 gives no obstruction.

Nonetheless, if we take any smooth curve of SKT Hermitian invariant metrics $\{\omega_t\}$ along $\varphi(t)$ such that $\omega_0 = \omega$, written as $\omega_t = e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\omega(t))$ with

$$\omega(t) = \frac{i}{2}\sum_{j=1}^{4} a_{j\bar{j}}(t)\, \eta^{j\bar{j}} + \frac{1}{2}\sum_{j<k}\left(\alpha_{j\bar{k}}(t)\, \eta^{j\bar{k}} - \overline{\alpha}_{j\bar{k}}(t)\, \eta^{k\bar{j}}\right),$$

and we compute

$$\partial\overline{\partial}\omega'(0) = \frac{1}{8}\mathfrak{Im}(\alpha'_{3\bar{4}}(0))\eta^{12\bar{1}\bar{2}},$$

by imposing equation (2.2.1) of Theorem 2.2.1 we obtain the following result.

Theorem 2.2.4 ([118]). *Let (M, J, g, ω) be the nilmanifold obtained as the compact quotient $\Gamma\backslash G$ of the Lie group $G := \mathbb{H}(3;\mathbb{R}) \times \mathbb{H}(3;\mathbb{R}) \times \mathbb{R}^2$ by a lattice Γ of G, with complex structure J defined through the invariant coframe of $(1,0)$-complex forms $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ with structure equations*

$$\begin{cases} d\eta^1 = 0, \quad d\eta^2 = 0, \\ d\eta^3 = -\frac{i}{2}\eta^{1\bar{1}}, \\ d\eta^4 = -\frac{i}{2}\eta^{2\bar{2}}. \end{cases}$$

Let us consider the curve of deformations

$$t \mapsto \varphi(t) = t(a_{11}\overline{\eta}^1 \otimes Z_1 + a_{22}\overline{\eta}^2 \otimes Z_2 + a_{32}\overline{\eta}^2 \otimes Z_3 + a_{33}\overline{\eta}^3 \otimes Z_3 + \\ + a_{34}\overline{\eta}^4 \otimes Z_3 + a_{41}\overline{\eta}^1 \otimes Z_4 + a_{43}\overline{\eta}^3 \otimes Z_4 + a_{44}\overline{\eta}^4 \otimes Z_4), \quad t \in (-\epsilon, \epsilon)$$

and any smooth curve of Hermitian invariant metrics $\{\omega_t\}_{t\in(-\epsilon,\epsilon)}$ *along* $\varphi(t)$ *such that* $\omega_0 = \omega$*, with* $\omega_t = e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}(\omega(t))$*, where*

$$\omega(t) = \frac{i}{2}\sum_{j=1}^{4} a_{j\bar{j}}(t)\ \eta^{j\bar{j}} + \frac{1}{2}\sum_{j<k}\left(\alpha_{j\bar{k}}(t)\ \eta^{j\bar{k}} - \overline{\alpha}_{j\bar{k}}(t)\ \eta^{k\bar{j}}\right).$$

Then a necessary condition for ω_t *to be SKT for any* $t \in (-\epsilon,\epsilon)$ *is that*

$$i\alpha_{3\bar{3}}(a_{34} + \overline{a}_{34}) + i\alpha_{4\bar{4}}(a_{43} + \overline{a}_{43}) + \alpha_{3\bar{4}}(a_{33} + \overline{a}_{44}) - \overline{\alpha}_{3\bar{4}}(a_{44} + \overline{a}_{33}) = \mathfrak{Im}(\alpha'_{3\bar{4}}(0)).$$

2.3 Deformations of Astheno-Kähler metrics

Let (M, J, g, ω) be a compact complex manifold of complex dimension n endowed with an astheno-Kähler metric g, i.e., $\partial\overline{\partial}\omega^{n-2} = 0$ and let $\{M_t\}_{t\in I}$ be a differentiable family of deformations such that $M_0 = M$, with $\{M_t\}_{t\in I}$ parametrized by a $(0,1)$-vector form $\varphi(t)$ on M. Let also $\{\omega_t\}_{t\in I}$ be a family of Hermitian metrics on $\{M_t\}_{t\in U}$, such that $\omega_0 = \omega$ and suppose that g_t is balanced on M_t, for every $t \in I$, i.e,

$$\partial_t\overline{\partial}_t\omega_t^{n-2} = 0, \quad \forall t \in I. \tag{2.3.1}$$

By Lemma 1.7.6, we can write each ω_t as $e^{i_\varphi|i_{\overline{\varphi}}}\omega(t)$, with $\omega(t) = \omega_{ij}(t)dz^i \wedge d\overline{z}^j$, locally, and also

$$\omega_t^{n-2} = e^{i_\varphi|i_{\overline{\varphi}}}(\omega^{n-2}(t)) = e^{i_\varphi|i_{\overline{\varphi}}}\left(f_v(t)dz^{i_1} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge dz^{i_{n-2}} \wedge d\overline{z}^{j_{n-2}}\right),$$

where $f_v(t) := \omega_{i_1j_1}\dots\omega_{i_{n-2}j_{n-2}}$ and $v = (i_1, j_1, \dots, i_{n-2}, j_{n-2})$, and $i_k, j_k \in \{1,\dots,n\}$, $k \in \{1,\dots,n-2\}$.

We can then apply formulas (2.1.2) and (2.1.3) to (2.3.1), and by expanding in Taylor series and differentiating with respect to t in $t = 0$, we are able to prove the following theorem.

Theorem 2.3.1. *Let* (M,J) *be a* n*-dimensional compact complex manifold endowed with an astheno-Kähler metric* g *and associated fundamental form* ω*. Let* $\{M_t\}_{t\in I}$ *be a differentiable family of compact complex manifolds with* $M_0 = M$ *and parametrized by* $\varphi(t) \in \mathcal{A}^{0,1}(T^{1,0}(M))$*, for* $t \in I := (-\epsilon,\epsilon)$*,* $\epsilon > 0$*. Let* $\{\omega_t\}_{t\in I}$ *be a smooth family of Hermitian metrics along* $\{M_t\}_{t\in I}$*, written as*

$$\omega_t = e^{i_\varphi|i_{\overline{\varphi}}}\ (\omega(t)),$$

where, locally, $\omega(t) = \omega_{ij}(t)\,dz^i \wedge d\overline{z}^j \in \mathcal{A}^{1,1}(M)$ *and* $\omega_0 = \omega$*.*

If ω_t^{n-2} *has local expression* $e^{i_\varphi|i_{\overline{\varphi}}}(\omega^{n-2}(t)) = e^{i_\varphi|i_{\overline{\varphi}}}(f_v(t)\,dz^{i_1} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge dz^{i_{n-2}} \wedge d\overline{z}^{j_{n-2}})$*, set*

$$(\omega^{n-2}(t))' := \frac{\partial}{\partial t}(f_v(t))\,dz^{i_1} \wedge d\overline{z}^{j_1} \wedge \dots dz^{i_{n-2}} \wedge d\overline{z}^{j_{n-2}} \in \mathcal{A}^{n-2,n-2}(M).$$

Then, if every metric ω_t *is astheno-Kähler, for* $t \in I$*, it must hold that*

$$2i\mathfrak{Im}(\partial \circ i_{\varphi'(0)} \circ \partial)(\omega^{n-2}) = \partial\overline{\partial}(\omega^{n-2}(0))'. \tag{2.3.2}$$

As a direct consequence, we immediately have the following corollary.

Corollary 2.3.2. *Let (M, J) be a compact Hermitian manifold endowed with an asteno-Kähler metric g and associated fundamental form ω. If there exists a smooth family of astheno-Kähler metrics which coincides with ω in $t = 0$, along the family of deformations $\{M_t\}_t$ with $M_0 = M$ and parametrized by the $(0,1)$-vector form $\varphi(t)$ on M, then the following equation must hold*

$$\left[(\partial \circ i_{\varphi'(0)} \circ \partial)(\omega^{n-2})\right]_{H^{n-1,n-1}_{BC}(M)} = 0. \tag{2.3.3}$$

Proof of Theorem 2.3.1. The metrics ω_t are astheno-Kähler for every $t \in I$, i.e., $\partial_t\overline{\partial}_t\omega_t^{n-2} = 0$. This implies

$$\frac{\partial}{\partial t}(\partial_t\overline{\partial}_t\omega_t^{n-2}) = 0. \tag{2.3.4}$$

Let us then compute the right hand side of (2.3.4) through formulas (2.1.2) and (2.1.3) for the operators ∂_t and $\overline{\partial}_t$. By the extension map we have that

$$\partial_t\overline{\partial}_t(\omega_t^{n-2}) = \partial_t\overline{\partial}_t(e^{i_\varphi|i_{\overline{\varphi}}}(\omega^{n-2}(t))),$$

and then, by (2.1.2) and (2.1.3), we have

$$\begin{aligned}
&\partial_t\overline{\partial}_t(e^{i_\varphi|i_{\overline{\varphi}}}(\omega^{n-2}(t)) = \partial_t(e^{i_\varphi|i_{\overline{\varphi}}}\left((I-\overline{\varphi}\varphi)^{-1}\lrcorner([\partial, i_\varphi]+\overline{\partial})(I-\overline{\varphi\varphi})\lrcorner\omega^{n-2}(t)\right))\\
&= e^{i_\varphi|i_{\overline{\varphi}}}\Bigg(\left((I-\varphi\overline{\varphi})^{-1}\lrcorner([\overline{\partial}, i_{\overline{\varphi}}]+\partial)(I-\varphi\overline{\varphi})\right)\lrcorner\Big((I-\overline{\varphi}\varphi)^{-1}\lrcorner([\partial, i_\varphi]+\overline{\partial})(I-\overline{\varphi\varphi})\lrcorner\omega^{n-2}(t)\Big)\Bigg).
\end{aligned}$$

Now, we expand in Taylor series centered in $t = 0$ the terms

$$\varphi(t) = t\varphi'(0) + o(t), \quad \omega^{n-2}(t) = \omega^{n-2}(0) + t\omega^{n-2}(0)' + o(t)$$

and recalling (2.1.1) and (2.2.3), we obtain that

$$\begin{aligned}
&\partial_t\overline{\partial}_t\omega_t^{n-2}\\
&= \left(I + t\varphi'(0) + \overline{t\varphi}'(0)\right)\lrcorner\left([\overline{\partial}, \overline{t\varphi'(0)\lrcorner}] + \partial\right)\left([\partial, t\varphi'(0)\lrcorner] + \overline{\partial}\right)\left(\omega^{n-2}(0) + t(\omega^{n-2}(0))'\right) + o(t)\\
&= \left(I + t\varphi'(0) + \overline{t\varphi}'(0)\right)\lrcorner\left([\overline{\partial}, \overline{t\varphi'(0)\lrcorner}] + \partial\right)\left([\partial, t\varphi'(0)\lrcorner]\omega^{n-2}(0) + \overline{\partial}\omega^{n-2}(0) + t\overline{\partial}(\omega^{n-2}(0))'\right) + o(t)\\
&= \left(I + t\varphi'(0) + \overline{t\varphi}'(0)\right)\lrcorner\left(-t\partial(\varphi'(0)\lrcorner\omega^{n-2}(0)) + t\overline{\partial}(\overline{\varphi'(0)}\lrcorner\overline{\partial}\omega^{n-2}(0)) + t\partial\overline{\partial}(\omega^{n-2}(0))'\right) + o(t)\\
&= -t\partial(\varphi'(0)\lrcorner\partial\omega^{n-2}(0)) + t\overline{\partial}(\overline{\varphi'(0)}\lrcorner\overline{\partial}\omega^{n-2}(0)) + t\partial\overline{\partial}(\omega^{n-2}(0))' + o(t).
\end{aligned}$$

Now, since $\partial_t\overline{\partial}_t\omega_t^{n-2} = 0$, for every $t \in I$, also, $\frac{\partial}{\partial t}\big|_{t=0}(\partial_t\overline{\partial}_t\omega_t^{n-2}) = 0$, hence

$$-\partial(\varphi'(0)\lrcorner\partial\omega^{n-2}(0)) + \overline{\partial}(\overline{\varphi'(0)}\lrcorner\overline{\partial}\omega^{n-2}(0)) + \partial\overline{\partial}(\omega^{n-2}(0))' = 0,$$

which is equivalent to

$$-(\partial \circ i_{\varphi'(0)} \circ \partial)\omega^{n-2} + (\overline{\partial} \circ i_{\overline{\varphi'(0)}} \circ \overline{\partial})\omega^{n-2} + \partial\overline{\partial}(\omega^{n-2}(0))' = 0,$$

hence, concluding the proof. □

As an application of Theorem 2.3.1 and Corollary 2.3.2, we provide examples of obstructions to the existence of curve of astheno-Kähler metrics on two families of 4-dimensional nilmanifolds.

2.3.1 Example 1

Let $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ be the coframe of invariant $(1,0)$-forms on the nilmanifold $(M = \Gamma\backslash G, J)$ of Example 2.2.1, with complex structure J defined by the structure equations

$$\begin{cases} d\eta^i & = 0, \quad i \in \{1,2,3\}, \\ d\eta^4 & = a_1\eta^{12} + a_2\eta^{13} + a_3\eta^{1\bar{1}} + a_4\eta^{1\bar{2}} + a_5\eta^{1\bar{3}} \\ & +a_6\eta^{23} + +a_7\eta^{2\bar{1}} + a_8\eta^{2\bar{2}} + a_9\eta^{2\bar{3}} \\ & +a_{10}\eta^{3\bar{1}} + a_{11}\eta^{3\bar{2}} + a_{12}\eta^{3\bar{3}}, \end{cases} \tag{2.3.5}$$

with $a_j \in \mathbb{Q}[i]$, for every $j \in \{1,2,\dots,12\}$. If $\omega = \frac{i}{2}\sum_{j=1}^4 \eta^{j\bar{j}}$ is the fundamental form associated to the diagonal metric g on M, then ω is astheno-Kähler i.e.,

$$\partial\bar{\partial}\omega^2 = 0$$

if, and only if,

$$|a_1|^2 + |a_2|^2 + |a_4|^2 + |a_5|^2 + |a_6|^2 + |a_7|^2 + |a_9|^2 + |a_{10}|^2 + |a_{11}|^2 = 2\mathfrak{Re}(a_3\bar{a}_8 + a_3\bar{a}_{12} + a_8\bar{a}_{12}).$$

Remark 2.3.3. If the complex manifold (M,J) is holomorphically parallelizable, i.e.,

$$a_3 = a_4 = a_5 = a_7 = a_8 = a_9 = a_9 = a_{10} = a_{11} = a_{12} = 0,$$

then metric g is astheno Kähler on (M,J) if, and only if, also $a_3 = a_8 = a_{12} = 0$, i.e., (M,J) is a complex torus. This is in line with the more general argument that on a compact holomorphically parallelizable manifold there exists a global coframe of holomorphic $(1,0)$-form; however, if a manifold admits an astheno-Kähler metric, every holomorphic 1-form is d-closed. Therefore, on a holomorphically parallelizable manifold endowed with an astheno-Kähler metric, each form of the global holomorphic coframe is d-closed, hence the manifold is a torus.

If $\{Z_1, Z_2, Z_3, Z_4\}$ is the dual frame of $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ on M, we can define the smooth $(0,1)$-vector form $\varphi(\mathbf{t})$ on M by

$$\varphi(\mathbf{t}) := t_1\bar{\eta}^1 \otimes Z_1 + t_2\bar{\eta}^2 \otimes Z_2 + t_3\bar{\eta}^3 \otimes Z_3, \qquad \mathbf{t} = (t_1,t_2,t_3) \in B := \{\mathbf{t} \in \mathbb{C}^3 : |t| < 1\} \tag{2.3.6}$$

which parametrizes a family of (non necessarily integrable) deformations $\{(M, J_t)\}_{t\in B}$ of (M,J). By Lemma 1.7.6, each (almost) complex structure J_t can be characterized by the coframe $\{\eta^1_{\mathbf{t}}, \eta^2_{\mathbf{t}}, \eta^3_{\mathbf{t}}, \eta^4_{\mathbf{t}}\}$ given by

$$\begin{cases} \eta^1_{\mathbf{t}} = \eta^1 + t_1\bar{\eta}^1 \\ \eta^2_{\mathbf{t}} = \eta^2 + t_2\bar{\eta}^2 \\ \eta^3_{\mathbf{t}} = \eta^3 + t_3\bar{\eta}^3 \\ \eta^4_{\mathbf{t}} = \eta^4, \end{cases} \tag{2.3.7}$$

which yields, by inverting the system,

$$\begin{cases} \eta^1 = \frac{1}{1-|t_1|^2}(\eta^1_{\mathbf{t}} - t_1\bar{\eta}^1_{\mathbf{t}}) \\ \eta^2 = \frac{1}{1-|t_2|^2}(\eta^2_{\mathbf{t}} - t_2\bar{\eta}^2_{\mathbf{t}}) \\ \eta^3 = \frac{1}{1-|t_3|^2}(\eta^3_{\mathbf{t}} - t_3\bar{\eta}^3_{\mathbf{t}}) \\ \eta^4 = \eta^4_{\mathbf{t}}. \end{cases} \tag{2.3.8}$$

Set $T_j := \frac{1}{1-|t_j|^2}$, for $j \in \{1,2,3\}$. By Remark 1.7.7, $\varphi(\mathbf{t})$ parametrizes a family of compact complex manifold, i.e., each J_t defines an integrable almost complex structure on M, if, and only if,

$$(d\eta^j_{\mathbf{t}})^{0,2} = 0, \qquad j \in \{1,2,3,4\}.$$

By relations (2.3.7), (2.3.8) and structure equations (2.3.5), it turns out that such an integrability condition holds if, and only if

$$\begin{aligned}(d\eta^4_{\mathbf{t}})^{0,2} &= T_1T_2(a_1t_1t_2 - a_4t_1 + a_7t_2)\eta^{\overline{12}}_{\mathbf{t}}\\ &\quad + T_1T_2(a_2t_1t_3 - a_5t_1 + a_{10}t_3)\eta^{\overline{13}}_{\mathbf{t}}\\ &\quad + T_2T_3(a_6t_2t_3 - a_9t_2 + a_{11}t_3)\eta^{\overline{23}}_{\mathbf{t}} = 0,\end{aligned}$$

i.e.,

$$\begin{cases} a_1t_1t_2 - a_4t_1 + a_7t_2 = 0\\ a_2t_1t_3 - a_5t_1 + a_{10}t_3 = 0\\ a_6t_2t_3 - a_9t_2 + a_{11}t_3 = 0.\end{cases} \tag{2.3.9}$$

Now let us fix a choice of $a_j \in \mathbb{Q}[i]$ and let $S \subset B$ be the set of solutions of system (2.3.9). Then, for every $\mathbf{t} \in S$, $\varphi(\mathbf{t})$ parametrizes the family of deformations $\{(M, J_{\mathbf{t}})\}_{\mathbf{t}\in S}$. Moreover, we can consider a curve $\gamma\colon(-\epsilon,\epsilon) \to S$, with $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \in S$, so that, for every $t \in (-\epsilon, \epsilon)$, the $(0,1)$-vector form

$$\varphi(\gamma(t)) = \varphi_1(t)\bar{\eta}^1 \otimes Z_1 + \varphi_2(t)\bar{\eta}^2 \otimes Z_2 + \varphi_3(t)\bar{\eta}^3 \otimes Z_3$$

parametrizes a curve of deformations. From now on, with an abuse of notation, we will write $\varphi(t) := \varphi(\gamma(t))$. We will have then

$$\varphi'(0) := \varphi_1'(0)\bar{\eta}^1 \otimes Z_1 + \varphi_2'(0)\bar{\eta}^2 \otimes Z_2 + \varphi_2'(0)\bar{\eta}^3 \otimes Z_3.$$

In order to apply condition (2.3.3), we observe the following facts regarding the form $(\partial \circ i_{\varphi'(0)} \circ \partial)\omega^2$ and the Bott-Chern cohomology of bidegree $(3,3)$ of M.

Lemma 2.3.4. *(I)* $(\partial \circ i_{\varphi'(0)} \circ \partial)\omega^2 = C_{J,\varphi'(0)}\eta^{123\overline{123}}$, *with* $C_{J,\varphi'(0)} \in \mathbb{Q}[i]$.

(II) If J is abelian, then $(\partial \circ i_{\varphi'(0)} \circ \partial)\omega^2 = 0$.

(III) The Bott-Chern cohomology class $[\eta^{123\overline{123}}]_{BC} \neq 0$ *if, and only if, ω is SKT.*

(IV) for every constant $C \in \mathbb{C}$, it holds $\mathfrak{Im}(C\eta^{123\overline{123}}) = -i\mathfrak{Re}(C)\eta^{123\overline{123}}$.

Proof. (I) By simple computations, it holds that

$$(-2)\omega^2 = \eta^{1\bar{1}2\bar{2}} + \eta^{1\bar{1}3\bar{3}} + \eta^{1\bar{1}4\bar{4}} + \eta^{2\bar{2}3\bar{3}} + \eta^{2\bar{2}4\bar{4}} + \eta^{3\bar{3}4\bar{4}}.$$

Let us rewrite structure equations (2.3.5) as

$$\begin{cases} d\eta^j = 0, \quad j \in \{1,2,3\},\\ d\eta^4 = \sum_{1\le i<j\le 3} A_{ij}\eta^{ij} + \sum_{i,j=1}^3 B_{i\bar{j}}\eta^{i\bar{j}}.\end{cases}$$

For the sake of completeness, we write

$$\partial\eta^4 = \sum_{1\le i<j\le 3} A_{ij}\eta^{ij}, \qquad \overline{\partial}\eta^4 = \sum_{i,j=1}^{3} B_{i\bar{j}}\eta^{i\bar{j}}$$

$$\partial\overline{\eta}^4 = -\sum_{i,j=1}^{3} \overline{B}_{i\bar{j}}\eta^{j\bar{i}}, \qquad \overline{\partial}\overline{\eta}^4 = \sum_{1\le i<j\le 3} \overline{A}_{ij}\eta^{\bar{i}\bar{j}}.$$

Then, it is easy to see that

$$\begin{aligned}(-2)\partial\omega^2 &= (\eta^{1\bar{1}} + \eta^{2\bar{2}} + \eta^{3\bar{3}}) \wedge \partial\eta^{4\bar{4}} \\ &= (\eta^{1\bar{1}} + \eta^{2\bar{2}} + \eta^{3\bar{3}}) \wedge (A_{ij}\eta^{ij\bar{4}} + \overline{B}_{i\bar{j}}\eta^{4j\bar{i}}) \\ &= (\eta^{1\bar{1}} + \eta^{2\bar{2}} + \eta^{3\bar{3}}) \wedge (A_{ij}\eta^{ij\bar{4}}) \\ &+ (\eta^{1\bar{1}} + \eta^{2\bar{2}} + \eta^{3\bar{3}}) \wedge (\overline{B}_{i\bar{j}}\eta^{4j\bar{i}})\end{aligned}$$

Now, since $\varphi'(0) = \varphi'_1(0)\overline{\eta}^1 \otimes Z_1 + \varphi'_2(0)\overline{\eta}^2 \otimes Z_2 + \varphi'_3(0)\overline{\eta}^3 \otimes Z_3$, we have that

$$\begin{aligned}(-2)(i_{\varphi'(0)} \circ \partial)\omega^2 &= (\eta^{1\bar{1}} + \eta^{2\bar{2}} + \eta^{3\bar{3}}) \wedge \left[A_{ij}\left(\sum_{k=1}^{3} \varphi'_k(0)\,\overline{\eta}^k \wedge i_{Z_k}(\eta^{ij})\right)\right] \wedge \overline{\eta}^4 \\ &- (\eta^{1\bar{1}} + \eta^{2\bar{2}} + \eta^{3\bar{3}}) \wedge \eta^4 \wedge \left[\overline{B}_{i\bar{j}}\left(\sum_{k=1}^{3} \varphi'_k(0)\,\overline{\eta}^k \wedge i_{Z_k}(\eta^{j\bar{i}})\right)\right].\end{aligned}$$

Note that the $(1,1)$-forms

$$\Omega_1 := A_{ij}\left(\sum_{k=1}^{3} \varphi'_k(0)\,\overline{\eta}^k \wedge i_{Z_k}(\eta^{ij})\right)$$

and

$$\Omega_2 := \overline{B}_{i\bar{j}}\left(\sum_{k=1}^{3} \varphi'_k(0)\,\overline{\eta}^k \wedge i_{Z_k}(\eta^{j\bar{i}})\right)$$

do not contain η^4 nor $\overline{\eta}^4$. Then,

$$\begin{aligned}(-2)(\partial \circ i_{\varphi'(0)}) \circ \partial)(\omega^2) &= (\eta^{1\bar{1}} + \eta^{2\bar{2}} + \eta^{3\bar{3}}) \wedge \Omega_1 \wedge \partial\overline{\eta}^4 \\ &+ (\eta^{1\bar{1}} + \eta^{2\bar{2}} + \eta^{3\bar{3}}) \wedge \partial\eta^4 \wedge \Omega_2 \\ &= -(\eta^{1\bar{1}} + \eta^{2\bar{2}} + \eta^{3\bar{3}}) \wedge \Omega_1 \wedge \left(\overline{B}_{i\bar{j}}\eta^{j\bar{i}}\right) \\ &+ (\eta^{1\bar{1}} + \eta^{2\bar{2}} + \eta^{3\bar{3}}) \wedge \left(A_{ij}\eta^{ij}\right) \wedge \Omega_2,\end{aligned}$$

i.e, a $(3,3)$-form on M which does not contain η^4 nor $\overline{\eta}^4$. Hence, $(\partial \circ i_{\varphi'(0)} \circ \partial)\omega^2 = C_{J,\varphi}\eta^{123\overline{123}}$.

(II) If J is abelian, then $A_{ij} = 0$, for every i, j with $1 \le i < j \le 3$. Then, by the previous point, $\Omega_1 = 0$ and clearly $(\partial \circ i_{\varphi'(0)} \circ \partial)\omega^2 = 0$.

(III) Let us assume that the metric ω is SKT, i.e., $\partial\overline{\partial}\omega = 0$. By structure equations (2.3.5) this is equivalent to $\partial\overline{\partial}\eta^{4\bar{4}} = 0$. Since the form $\eta^{123\overline{123}}$ is d-closed, the Bott-Chern class $[\eta^{123\overline{123}}]$ is well defined. Moreover, since $\partial\overline{\partial} *_g \eta^{123\overline{123}} = \partial\overline{\partial}\eta^{4\bar{4}} = 0$, the form $\eta^{123\overline{123}}$ is Bott-Chern harmonic, hence $[\eta^{123\overline{123}}]_{BC} \neq 0$.
Viceversa, let us assume that ω is not SKT, i.e, $\partial\overline{\partial}\omega \neq 0$, which, by structure equations, is equivalent to $\partial\overline{\partial}\eta^{4\bar{4}} \neq 0$. Note that $\partial\overline{\partial}\eta^{4\bar{4}}$ is $(2,2)$-form on M, hence $\partial\overline{\partial}\eta^{4\bar{4}} = \sum_{i<j,k<l} A_{ij\overline{kl}}\eta^{ij\overline{kl}}$, with $i, j, k, \in$

$\{1,2,3\}$. Then, one can choose an invariant $(1,1)$-form α on M such that it does contain η^4 nor $\overline{\eta}^4$ and $\partial\overline{\partial}\eta^{4\overline{4}} \wedge \alpha = C\eta^{123\overline{123}} \neq 0$, $C \in \mathbb{C}$. In particular, $\partial\overline{\partial}\alpha = 0$. But then,

$$\partial\overline{\partial}\left(\frac{1}{C}\eta^{4\overline{4}} \wedge \alpha\right) = \frac{1}{C}\partial\overline{\partial}(\eta^{4\overline{4}}) \wedge \alpha = \eta^{123\overline{123}},$$

which implies that the form $\eta^{123\overline{123}}$ is $\partial\overline{\partial}$-exact. Therefore $[\eta^{123\overline{123}}]_{BC} = 0$.

(IV) For every $C \in \mathbb{C}$, one has that $\mathfrak{Im}(C\eta^{123\overline{123}}) = \frac{1}{2i}\left(C\eta^{123\overline{123}} - \overline{C}\eta^{\overline{123}123}\right) = \frac{1}{2i}(C + \overline{C})\eta^{123\overline{123}} = -i\mathfrak{Re}(C)\eta^{123\overline{123}}$. □

As a consequence of Lemma 2.3.4 and Remark 2.3.3, it turns out that Corollary 2.3.2 can yield obstructions on this family of 4-dimensional nilmanifolds and on the curve of deformations of a fixed element (M,J) of the family parametrized by $\varphi(\gamma(t))$, only if the starting complex structure J is not holomorphically parallelizable nor abelian and the starting diagonal metric on (M,J) is both astheno-Kähler and SKT.

Remark 2.3.5. Notice that arguments similar to Lemma 2.3.4 and Remark 2.3.3 are still valid on any n-dimensional nilmanifold (M,J) with left-invariant complex structure J characterized by analogous structure equations, i.e., when (M,J) is a nilmanifold whose complex structure is determined by a coframe of left-invariant $(1,0)$-forms $\{\eta^1,\dots,\eta^n\}$ such that

$$\begin{cases} d\eta^i = 0, \quad i \in \{1,\dots n-1\}, \\ d\eta^n \in \mathrm{Span}_{\mathbb{C}}\{\eta^{ij}, \eta^{k\bar{l}}\}, \quad i,j,k,l \in \{1,\dots,n-1\}, \end{cases}$$

and whose coefficient structures are elements of $\mathbb{Q}[i]$. More specifically, if $\omega = \frac{i}{2}\sum_{j=1}^n \eta^{j\bar{j}}$ is the fundamental form associated to the diagonal metric g and $\varphi(t) = \sum_{k=1}^{n-1}\varphi_k(t)\overline{\eta}^k \otimes Z_k$, for $t \in (\epsilon,\epsilon)$ is a curve of deformations of (M,J), then the following holds.

Lemma 2.3.6. *(I) $(\partial \circ i_{\varphi'(0)} \circ \partial)\omega^{n-2} = C_{J,\varphi}\,\eta^{1\dots n-1\,\overline{1}\dots\overline{n-1}}$.*
(II) If J is abelian, then $(\partial \circ i_{\varphi'(0)} \circ \partial)\omega^{n-2} = 0$. (III) The Bott-Chern cohomology class $[\eta^{1\dots n-1\,\overline{1}\dots\overline{n-1}}] \neq 0$ if and only if ω is SKT.
(IV) if n is even, for every constant $C \in \mathbb{C}$, $\mathfrak{Im}(C\eta^{1\dots n-1\,\overline{1}\dots\overline{n-1}}) = -i\mathfrak{Re}(C)\eta^{1\dots n-1\,\overline{1}\dots\overline{n-1}}$.
(V) if n is odd, for every constant $c \in \mathbb{C}$, $\mathfrak{Im}(C\eta^{1\dots n-1\,\overline{1}\dots\overline{n-1}}) = \mathfrak{Im}(C)\eta^{1\dots n-1\,\overline{1}\dots\overline{n-1}}$.

Therefore, as in the 4-dimensional case Corollary 3.2.4 could yield obstructions to the existence of curves of astheno-Kähler metrics along the family of deformations parametrized by φ on such families of nilmanifolds of complex dimension n only if the canonical diagonal metric g is also SKT and the complex structure J is nor abelian nor holomorphically parallelizable.

Let us go back the 4-dimensional family and consider an element (M,J) of the family of 4-dimensional nilmanifolds, with $a_2 = a_5 = a_6 = a_9 = a_{10} = a_{11} = a_{12} = 0$. In particular, we are annihilating the coefficients of the second and third row of (2.3.9). By the symmetry of the system, one obtains similar results by annihilating either the first and third rows, or the second and third row. In this case, structure equations become

$$\begin{cases} d\eta^i &= 0, \quad i \in \{1,2,3\}, \\ d\eta^4 &= a_1\eta^{12} + a_3\eta^{1\overline{1}} + a_4\eta^{1\overline{2}} \\ & \quad + a_7\eta^{2\overline{1}} + a_8\eta^{2\overline{2}}. \end{cases} \tag{2.3.10}$$

The diagonal metric g with fundamental form $\omega = \frac{i}{2}\sum_{j=1}^{4}\eta^{j\bar{j}}$ is astheno-Kähler if, and only if, is SKT if, and only if,

$$|a_1|^2 + |a_4|^2 + |a_7|^2 = 2\mathfrak{Re}(a_3\bar{a}_8). \tag{2.3.11}$$

On (M, J) we consider the $(0,1)$-vector form $\varphi(\mathbf{t})$, $t \in B$, as in (2.3.6). Such a vector form parametrizes a family of deformations (M, J_t) of (M, J) and each J_t is integrable if $\mathbf{t} = (t_1, t_2, t_3)$ satisfies

$$a_1t_1t_2 - a_4t_1 + a_7t_2 = 0. \tag{2.3.12}$$

Let us set $F(t_1, t_2, t_3) := a_1t_1t_2 - a_4t_1 + a_7t_2$, $(t_1, t_{2,3}) \in B$. Then, the gradient of F in $(0,0,0)$ is

$$\nabla F|_{(0,0,0)} = \begin{pmatrix} -a_4 \\ a_7 \\ 0 \end{pmatrix}.$$

By distinguishing the cases in which either $\nabla F|_{(0,0,0)} = 0$ or $\nabla F|_{(0,0,0)} \neq 0$, we obtain the following.

Case (i): $\nabla F|_{(0,0,0)} = 0$.

In this case, it holds that $a_4 = a_7 = 0$. Hence the only non zero differential of (2.3.10) becomes

$$d\eta^4 = a_1\eta^{12} + a_3\eta^{1\bar{1}} + a_8\eta^{2\bar{2}},$$

and let assume that ω is astheno-Kähler (and SKT), i.e., it holds

$$|a_1|^2 = 2\mathfrak{Re}(a_3\bar{a}_8).$$

We will assume $a_1 \neq 0$.

The solution set of (2.3.12) is

$$S = \{(t_1, t_2, t_3) \in B : t_1t_2 = 0\}.$$

Therefore, the $(0,1)$-vector form $\varphi(\mathbf{t})$ parametryzing the integrable deformations of M is

$$\varphi(\mathbf{t}) = t_1\bar{\eta}^1 \otimes Z_1 + t_2\bar{\eta}^2 \otimes Z_2 + t_3\bar{\eta}^3 \otimes Z_3, \quad (t_1, t_2, t_3) \in S,$$

and we can then consider the curve

$$\varphi(t) = t \cdot u_1\bar{\eta}^1 \otimes Z_1 + t \cdot u_2\bar{\eta}^2 \otimes Z_2 + t \cdot u_3\bar{\eta}^3 \otimes Z_3, \quad t \in (-\epsilon, \epsilon), \ \epsilon > 0,$$

for a fixed $(u_1, u_2, u_3) \in \mathbb{C}^3$. Hence,

$$\varphi'(0) = u_1\bar{\eta}^1 \otimes Z_1 + u_2\bar{\eta}^2 \otimes Z_2 + u_3\bar{\eta}^3 \otimes Z_3.$$

We can then apply our condition and compute

$$\partial \circ i_{\varphi'(0)} \circ \partial\omega^2 = 0,$$

hence neither Corollary 2.3.2 nor Theorem 2.3.1 yield obstructions.

Case (ii): $\nabla F|_{(0,0,0)} \neq 0$.

In this case, it holds that either $a_4 \neq 0$ or $a_7 \neq 0$. Suppose that $a_4 \neq 0$.

Then, structure equations (2.3.10) and the astheno-Kähler (and SKT) condition for ω (2.3.11) still holds. Then solution set of (2.3.12) is then

$$S = \left\{(t_1, t_2, t_3) \in \mathbb{C}^3 : t_1 = \frac{a_7 t_2}{a_4 - a_1 t_2}, |t_1| < \delta, |t_2| < \delta', |t_3| < \delta''\right\},$$

for $\delta, \delta', \delta'' > 0$ sufficiently small.

Hence the $(0,1)$-vector form $\varphi(\mathbf{t})$ parametrizing the integrable deformations of (M, J) is

$$\varphi(\mathbf{t}) = \frac{a_7 t_2}{a_4 - a_1 t_2}\overline{\eta}^1 \otimes Z_1 + t_2\overline{\eta}^2 \otimes Z_2 + t_3\overline{\eta}^3 \otimes Z_3, \quad \left(\frac{a_7 t_2}{a_4 - a_1 t_2}, t_2, t_3\right) \in S,$$

so that can consider the curve of deformations

$$\varphi(t) = \frac{t a_7 u_2}{a_4 - t a_1 u_2}\overline{\eta}^1 \otimes Z_1 + t u_2\overline{\eta}^2 \otimes Z_2 + t u_3\overline{\eta}^3 \otimes Z_3, \quad (u_2, u_3) \in \mathbb{C}^2,\ t \in (-\epsilon, \epsilon),$$

for $\epsilon > 0$ sufficiently small, so that

$$\varphi'(0) = \frac{a_7 u_2}{a_4}\overline{\eta}^1 \otimes Z_1 + u_2\overline{\eta}^2 \otimes Z_2 + u_3 \otimes Z_3.$$

By computations, we obtain that

$$\partial \circ i_{\varphi'(0)} \circ \partial\omega^2 = 2(|a_7|^2 - |a_4|^2)\frac{a_1 u_2}{a_4}\eta^{123\overline{123}}.$$

Therefore, since the Bott-Chern cohomology class $[\eta^{123\overline{123}}] \neq 0$ by Lemma 2.3.4, condition (2.3.3) holds if and only if

$$\left(|a_4|^2 - |a_7|^2\right)\mathfrak{Re}\left(\frac{a_1 u_2}{a_4}\right) = 0.$$

If $a_7 \neq 0$, we have that

$$S = \left\{(t_1, t_2, t_3) \in B : t_2 = \frac{a_4 t_1}{a_7 + a_1 t_1},\ |t_1| < \delta, |t_2| < \delta', |t_3| < \delta''\right\},$$

for $\delta, \delta', \delta'' > 0$ sufficiently small.

The curve of deformations hence is

$$\varphi(t) = t u_1\overline{\eta}^1 \otimes Z_1 + \frac{t a_4 u_1}{a_7 + t a_1 u_1}\overline{\eta}^2 \otimes Z_2 + t u_3\overline{\eta}^3 \otimes Z_3, \quad (u_1, u_3) \in \mathbb{C}^2,\ t \in (-\epsilon, \epsilon),$$

for $\epsilon > 0$ sufficiently small, so that

$$\varphi'(0) = u_1\overline{\eta}^1 \otimes Z_1 + \frac{a_4 u_1}{a_7}\overline{\eta}^2 \otimes Z_2 + u_3\overline{\eta}^3 \otimes Z_3.$$

Then, by computations similar to the previous case we obtain that

$$[\mathfrak{Im}(\partial \circ i_{\varphi'(0)} \circ \partial(\omega^2))]_{BC} = 0 \iff \left(|a_4|^2 - |a_7|^2\right)\mathfrak{Re}\left(\frac{a_1 u_1}{a_7}\right) = 0.$$

By applying Corollary 2.3.2 to each case, we obtain the following theorem.

Theorem 2.3.7. *Let $(M = \Gamma\backslash G, J)$ be an element of the family of 4-dimensional nilmanifolds with complex structure J determined by the coframe of left-invariant forms on G $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ with structure equations*

$$\begin{cases} d\eta^i = 0, \quad i \in \{1,2,3\} \\ d\eta^4 = a_1\eta^{12} + a_3\eta^{1\bar{1}} + a_4\eta^{1\bar{2}} + a_7\eta^{2\bar{1}} + a_8\eta^{2\bar{2}}, \end{cases}$$

with $a_1, a_3, a_4, a_7, a_8 \in \mathbb{Q}[i]$. Let $\omega = \frac{i}{2}\sum_{j=1}^4 \eta^{j\bar{j}}$ be the fundamental form associated to the diagonal metric, which we assume to be astheno-Kähler, i.e.,

$$|a_1|^2 + |a_4|^2 + |a_7|^2 = 2\mathfrak{Re}(a_3\bar{a}_8).$$

Then,

- *if $a_4 \neq 0$ and $(u_2, u_3) \in \mathbb{C}^2$, there exists no curve of astheno-Kähler metrics ω_t such that $\omega_0 = \omega$ along the curve of deformations $t \mapsto \varphi(t) = \frac{ta_7u_2}{a_4 - ta_1u_2}\bar{\eta}^1 \otimes Z_1 + tu_2\bar{\eta}^2 \otimes Z_2 + tu_3\bar{\eta}^3 \otimes Z_3$, for $t \in (-\epsilon, \epsilon)$ if*
$$\left(|a_4|^2 - |a_7|^2\right)\mathfrak{Re}\left(\frac{a_1u_2}{a_4}\right) \neq 0.$$
- *if $a_7 \neq 0$ and $(u_1, u_3) \in \mathbb{C}^2$, there exists no curve of astheno-Kähler metrics ω_t such that $\omega_0 = \omega$ along the curve of deformations $t \mapsto \varphi(t) = u_1\bar{\eta}^1 \otimes Z_1 + \frac{ta_4u_1}{a_7 + ta_1u_1}\bar{\eta}^2 \otimes Z_2 + tu_3\bar{\eta}^3 \otimes Z_3$, for $t \in (-\epsilon, \epsilon)$, if*
$$\left(|a_4|^2 - |a_7|^2\right)\mathfrak{Re}\left(\frac{a_1u_1}{a_7}\right) \neq 0.$$

2.3.2 Example 2

We now show an application of Corollary 2.3.2 to a 4-dimensional 2-step nilmanifold with invariant complex structure. Let (M, J) be the nilmanifold with $M = \Gamma\backslash G$ is the quotient of a nilpotent Lie group G by a discrete uniform subgroup Γ and the complex structure J determined by the coframe $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ of left-invariant $(1,0)$-forms on G with structure equations

$$\begin{cases} d\eta^1 = 0, \\ d\eta^2 = 0, \\ d\eta^3 = a_1\eta^{12} + a_2\eta^{1\bar{1}} + a_3\eta^{1\bar{2}} + a_4\eta^{2\bar{1}} + a_5\eta^{2\bar{2}} \\ d\eta^4 = b_1\eta^{12} + b_2\eta^{1\bar{1}} + b_3\eta^{1\bar{2}} + b_4\eta^{2\bar{1}} + b_5\eta^{2\bar{2}} \end{cases}$$

with $a_j, b_j \in \mathbb{Q}[i]$. Let $\omega = \sum_{j=1}^4 \eta^{j\bar{j}}$ be the fundamental form associated to the diagonal metric g. By computations, it turns out that g is astheno-Kähler if and only if

$$\begin{cases} 2\mathfrak{Re}(b_5\bar{b}_2) - |b_1|^2 - |b_3|^2 - |b_4|^2 = 0 \\ |b_1|^2 + |b_3|^2 - |b_4|^2 - b_5\bar{a}_2 - b_2\bar{a}_5 = 0 \\ 2\mathfrak{Re}(a_5\bar{a}_2) - |b_1|^2 - |b_3|^2 - |b_4|^2 = 0. \end{cases} \tag{2.3.13}$$

Since $\partial\bar{\partial}\omega = 0$ if and only if

$$\mathfrak{Re}(a_5\bar{a}_2) + \mathfrak{Re}(b_5\bar{b}_2) - |b_1|^2 - |b_3|^2 - |b_4|^2 = 0,$$

if the metric g is astheno-Kähler, it is also SKT.

Let us consider the family of deformations $(M, J_{\mathbf{t}})_{t\in B}$ of (M, J) parametrized by the $(0,1)$-vector form

$$\varphi(\mathbf{t}) = t_1\overline{\eta}^1 \otimes Z_1 + t_2\overline{\eta}^2 \otimes Z_2,$$

with $\mathbf{t} = (t_1, t_2) \in B := \{\mathbf{t} \in \mathbb{C}^2 : |\mathbf{t}| < \epsilon\}$, $\epsilon > 0$. The coframe $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ then changes under $\varphi(\mathbf{t})$ as

$$\begin{cases} \eta^1_{\mathbf{t}} = \eta^1 - t_1\overline{\eta}^1 \\ \eta^2_{\mathbf{t}} = \eta^2 - t_2\overline{\eta}^2 \\ \eta^3_{\mathbf{t}} = \eta^3 \\ \eta^4_{\mathbf{t}} = \eta^4 \end{cases}$$

so that, by reversing the system, we obtain

$$\begin{cases} \eta^1 = \frac{1}{1-|t_1|^2}(\eta^1_{\mathbf{t}} - t_1\overline{\eta}^1_{\mathbf{t}}) \\ \eta^2 = \frac{1}{1-|t_2|^2}(\eta^2_{\mathbf{t}} - t_2\overline{\eta}^2_{\mathbf{t}}) \\ \eta^3 = \eta^3_{\mathbf{t}} \\ \eta^4 = \eta^4_{\mathbf{t}}. \end{cases}$$

Set $T_j := \frac{1}{1-|t_j|^2}$, for $j \in \{frm[o]--, 2\}$.

Since the form $\varphi(\mathbf{t})$ defines an family of complex manifolds if and only if $d(\eta^j_{\mathbf{t}})^{0,2} = 0$, for every $j \in \{1, 2, 3, 4\}$, such a integrability condition is satisfied if and only if $(d\eta^3_{\mathbf{t}})^{0,2} = 0$, which yields

$$T_1T_2(a_1t_1t_2 - a_3t_1 + a_4t_2)\eta^{\overline{12}}_{\mathbf{t}} = 0$$

and $(d\eta^4_{\mathbf{t}})^{0,2} = 0$, which yields

$$T_1T_2(b_1t_1t_2 - b_3t_1 + b_4t_2)\eta^{\overline{12}}_{\mathbf{t}} = 0.$$

Under the assumption that $a_1 = b_1$, $a_3 = b_3$, and $a_4 = b_4$, we have that the condition of integrability is valid for $t \in S$, where S is the solution set of the equation

$$b_1t_1t_2 - b_3t_1 + b_4t_2 = 0, \quad (t_1, t_2, t_3) \in B. \tag{2.3.14}$$

We now proceed as in the usual manner, by considering the map

$$F(t_1, t_2) = b_1t_1t_2 - b_3t_1 + b_4t_2$$

and discussing the cases in which either $\nabla F|_{(0,0)} = \begin{pmatrix} -b_3 \\ b_4 \end{pmatrix}$ vanishes or not.

Case (i): $\nabla F|_{(0,0)} = 0$

This is the situation in which $b_3 = b_4 = 0$. Then structure equations become

$$\begin{cases} d\eta^1 = 0, \\ d\eta^2 = 0, \\ d\eta^3 = b_1\eta^{12} + a_2\eta^{1\overline{1}} + a_5\eta^{2\overline{2}} \\ d\eta^4 = b_1\eta^{12} + b_2\eta^{1\overline{1}} + b_5\eta^{2\overline{2}} \end{cases}$$

and the diagonal metric g is astheno-Kähler if, and only if, the following condition holds

$$\begin{cases} 2\mathfrak{Re}(b_5\overline{b}_2) = |b_1|^2 \\ b_2\overline{a}_5 + b_5\overline{a}_2 = |b_1|^2 \\ 2\mathfrak{Re}(a_5\overline{a}_2) = |b_1|^2 \end{cases}$$

We assume that $b_1 \neq 0$. The solution set S for equation (2.3.14) is then

$$S = \{(t_1, t_2) \in B : t_1 t_2 = 0\}$$

Hence as a curve of deformation $\varphi(t)$ with $t \in S$ we can choose

$$\varphi(t) = tu_1\overline{\eta}^1 \otimes Z_1 + tu_2\overline{\eta}^2 \otimes Z_2, \quad (u_1, u_2) \in S,\ t \in (-\delta, \delta)$$

for $\delta > 0$ sufficiently small. Then,

$$\varphi'(0) = u_1\overline{\eta}^1 \otimes Z_1 + u_2\overline{\eta}^2 \otimes Z_2.$$

By computations, however, it turns out that

$$\partial \circ i_{\varphi'(0)} \circ \partial\omega^2 = 0,$$

hence Corollary 2.3.2 does not yield any obstruction.

Case (ii): $\nabla F|_{(0,0)} \neq 0$.

In this situation, either $b_3 \neq 0$ or $b_4 \neq 0$. Let us assume $b_3 \neq 0$; the latter case is completely analogous.

We have the following structure equations

$$\begin{cases} d\eta^1 = 0 \\ d\eta^2 = 0 \\ d\eta^3 = b_1\eta^{12} + a_2\eta^{1\overline{1}} + b_3\eta^{1\overline{2}} + b_4\eta^{2\overline{1}} + a_5\eta^{2\overline{2}} \\ d\eta^4 = b_1\eta^{12} + b_2\eta^{1\overline{1}} + b_3\eta^{1\overline{2}} + b_4\eta^{2\overline{1}} + b_5\eta^{2\overline{2}} \end{cases}$$

and the astheno-Kähler condition (2.3.13) on the diagonal metric g is still valid. The solution set S for (2.3.14) is then

$$S = \{(t_1, t_2) \in B\, :\, t_1 = \frac{b_4 t_2}{b_3 - b_1 t_2}, |t_1| < \delta, |t_2| < \delta'\}$$

for $\delta, \delta' > 0$ sufficiently small. Once we fix $u_2 \in \mathbb{C}$, we can pick the curve of deformations

$$\varphi(t) = \frac{tb_4u_2}{b_3 - tb_1u_2}\overline{\eta}^1 \otimes Z_1 + tu_2\overline{\eta}^2 \otimes Z_2, \quad t \in (-\epsilon, \epsilon),$$

for $\epsilon > 0$ sufficiently small, so that

$$\varphi'(0) = \frac{b_4u_2}{b_3}\overline{\eta}^1 \otimes Z_1 + u_2\overline{\eta}^2 \otimes Z_2.$$

Then, we compute

$$(\partial \circ i_{\varphi'(0)} \circ \partial)\omega^2 = ((|b_3|^2 - |b_4|^2)\frac{b_1}{b_4}u_2)\eta^{123\overline{123}} + ((|b_4|^2 - |b_3|^2)\frac{b_1}{b_4}u_2)\eta^{123\overline{124}}$$

$$+((|b_4|^2-|b_3|^2)\frac{b_1}{b_4}u_2)\eta^{124\overline{123}}+((|b_3|^2-|b_4|^2)\frac{b_1}{b_4}u_2)\eta^{124\overline{124}}$$

Now, the forms $\eta^{123\overline{123}},\eta^{123\overline{124}},\eta^{124\overline{123}},\eta^{124\overline{124}}$ are all d-closed by structure equations. Moreover, by considering the $\mathbb{C}$-antilinear $*$-operator with respect to g, we see that

$$\begin{aligned}
\partial\overline{\partial}*\eta^{123\overline{123}}=0 &\iff 2\mathfrak{Re}(b_5\overline{b}_2)-|b_1|^2-|b_3|^2-|b_4|^2=0\\
\partial\overline{\partial}*\eta^{123\overline{124}}=0 &\iff |b_1|^2+|b_3|^2+|b_4|^2-b_5\overline{a}_2-b_2\overline{a}_5=0\\
\partial\overline{\partial}*\eta^{124\overline{123}}=0 &\iff |b_1|^2+|b_3|^2+|b_4|^2-a_2\overline{b}_5-a_5\overline{b}_2=0\\
\partial\overline{\partial}*\eta^{124\overline{124}}=0 &\iff 2\mathfrak{Re}(a_5\overline{a}_2)-|b_1|^2-|b_3|^2-|b_4|^2=0
\end{aligned}$$

Since ω is astheno-Kähler, i.e., conditions (2.3.13) hold, hence $\partial\overline{\partial}*\eta^{123\overline{123}}=\partial\overline{\partial}*\eta^{123\overline{124}}=\partial\overline{\partial}*\eta^{124\overline{123}}=\partial\overline{\partial}*\eta^{124\overline{124}}=0$, i.e., the forms $\eta^{123\overline{123}}$, $\eta^{123\overline{124}}$, $\eta^{124\overline{123}}$, $\eta^{124\overline{124}}$. Therefore,

$$\begin{aligned}
[\mathfrak{Im}(\partial\circ i_{\varphi'(0)}\circ\partial\omega^2)]_{BC}=&-i(|b_3|^2-|b_4|^2)\mathfrak{Re}(\frac{b_1u_2}{b_4})[\eta^{123\overline{123}}]+i(|b_3|^2-|b_4|^2)\mathfrak{Re}(\frac{b_1u_2}{b_4})[\eta^{123\overline{124}}]\\
&+i(|b_3|^2-|b_4|^2)\mathfrak{Re}(\frac{b_1u_2}{b_4})[\eta^{124\overline{123}}]-i(|b_3|^2-|b_4|^2)\mathfrak{Re}(\frac{b_1u_2}{b_4})[\eta^{124\overline{124}}]
\end{aligned}$$

which vanishes in $H^{3,3}_{BC}(M)$ if, and only if,

$$(|b_3|^2-|b_4|^2)\mathfrak{Re}(\frac{b_1u_2}{b_4})=0.$$

We summarise what we obtained in the following theorem.

Theorem 2.3.8. *Let (M,J) be an element of the family of 4-dimensional manifolds determined by the coframe of left-invariant $(1,0)$-forms $\{\eta^1,\eta^2,\eta^3,\eta^4\}$ with structure equations*

$$\begin{cases}
d\eta^1=0\\
d\eta^2=0\\
d\eta^3=b_1\eta^{12}+a_2\eta^{1\overline{1}}+b_3\eta^{1\overline{2}}+b_4\eta^{2\overline{1}}+a_5\eta^{2\overline{2}}\\
d\eta^4=b_1\eta^{12}+b_2\eta^{1\overline{1}}+b_3\eta^{1\overline{2}}+b_4\eta^{2\overline{1}}+a_5\eta^{2\overline{2}},
\end{cases}$$

with $a_2,a_5,b_1,b_2,b_3,b_4,b_5\in\mathbb{Q}[i]$. Let $\omega=\frac{i}{2}\sum_{j=1}^4\eta^{j\overline{j}}$ be the fundamental form associated to the diagonal metric g and suppose that g is astheno-Kähler (and hence SKT), i.e.,

$$\begin{cases}
2\mathfrak{Re}(b_5\overline{b}_2)-|b_1|^2-|b_3|^2-|b_4|^2=0\\
|b_1|^2+|b_3|^2-|b_4|^2-b_5\overline{a}_2-b_2\overline{a}_5=0\\
2\mathfrak{Re}(a_5\overline{a}_2)-|b_1|^2-|b_3|^2-|b_4|^2=0.
\end{cases}$$

Then,

- *if $b_3\neq 0$ and $u_2\in\mathbb{C}$, there exists no curve of astheno-Kähler metrics ω_t with $\omega_0=\omega$ along the curve of deformations $\varphi(t)=\frac{tb_4u_2}{b_3-tb_1u_2}\overline{\eta}^1\otimes Z_1+tu_2\overline{\eta}^2\otimes Z_2$, $t\in(-\epsilon,\epsilon)$, if*

$$(|b_3|^2-|b_4|^2)\mathfrak{Re}(\frac{b_1u_2}{b_4})\neq 0.$$

- *if $b_4\neq 0$ and $u_1\in\mathbb{C}$, there exists no curve of astheno-Kähler metrics ω_t with $\omega_0=\omega$ along the curve of deformations $\varphi(t)=tu_1\overline{\eta}^1\otimes Z_1+tu_2\frac{tb_3u_1}{tb_1u_1+b_4}\overline{\eta}^2\otimes Z_2$, $t\in(-\epsilon,\epsilon)$, if*

$$(|b_3|^2-|b_4|^2)\mathfrak{Re}(\frac{b_1u_1}{b_3})\neq 0.$$

2.4 Deformations of balanced metrics

Let now (M,J,g,ω) be a compact Hermitian manifold of complex dimension n endowed with a balanced metric g, i.e., $\overline{\partial}\omega^{n-1}=0$ and let $\{M_t\}_{t\in I}$ be a differentiable family of deformations such that $M_0=M$, with $\{M_t\}_{t\in I}$ parametrized by a $(0,1)$–vector form $\varphi(t)$ on M, for $t\in I=(-\epsilon,\epsilon)$, $\epsilon>0$. Let also $\{\omega_t\}_{t\in I}$ be a family of Hermitian metrics on $\{M_t\}_{t\in I}$, such that $\omega_0=\omega$ and we suppose the metrics g_t to be balanced, i.e.,

$$\overline{\partial}_t\omega_t^{n-1}=0,\qquad \forall t\in I. \tag{2.4.1}$$

We remark that, by Lemma 1.7.6, we can write each ω_t as $e^{i_\varphi|i_{\overline{\varphi}}}(\omega(t))$, where locally $\omega(t)=\omega_{ij}(t)dz^i\wedge d\overline{z}^j$. In particular, by definition of $e^{i_\varphi|i_{\overline{\varphi}}}$, it is easy to check that

$$\begin{aligned}\omega_t^{n-1}&=(e^{i_\varphi|i_{\overline{\varphi}}}(\omega(t)))^{n-1}=e^{i_\varphi|i_{\overline{\varphi}}}(\omega^{n-1}(t))\\&=e^{i_\varphi|i_{\overline{\varphi}}}(f_v(t)\,dz^{i_1}\wedge d\overline{z}^{j_1}\wedge\cdots\wedge dz^{i_{n-1}}\wedge d\overline{z}^{j_{n-1}}),\end{aligned}$$

where we denote $f_v(t):=\omega_{i_1j_1}(t)\ldots\omega_{i_{n-1}j_{n-1}}(t)$, with $v=(i_1,j_1,\ldots,i_{n-1},j_{n-1})$ and $i_k,j_k\in\{1,\ldots,n\}$, $k=\{1,\ldots,n-1\}$.

We can then apply formula (2.1.3) to (2.4.1) and, by making use of Taylor series expansion and differentiating with respect to t in $t=0$, we are able to prove the main theorem.

Theorem 2.4.1 ([130])**.** *Let (M,J) be a n-dimensional compact complex manifold endowed with a balanced metric g and associated fundamental form ω. Let $\{M_t\}_{t\in I}$ be a differentiable family of compact complex manifolds with $M_0=M$ and parametrized by $\varphi(t)\in\mathcal{A}^{0,1}(T^{1,0}(M))$, for $t\in I:=(-\epsilon,\epsilon)$, $\epsilon>0$. Let $\{\omega_t\}_{t\in I}$ be a smooth family of Hermitian metrics along $\{M_t\}_{t\in I}$, written as*

$$\omega_t=e^{i_\varphi|i_{\overline{\varphi}}}\,(\omega(t)),$$

where, locally, $\omega(t)=\omega_{ij}(t)\,dz^i\wedge d\overline{z}^j\in\mathcal{A}^{1,1}(M)$ and $\omega_0=\omega$.

If ω_t^{n-1} has local expression $e^{i_\varphi|i_{\overline{\varphi}}}(\omega_{i_1j_1}(t)\ldots\omega_{i_{n-1}j_{n-1}}(t)\,dz^{i_1}\wedge d\overline{z}^{j_1}\wedge\cdots\wedge dz^{i_{n-1}}\wedge d\overline{z}^{j_{n-1}})$, set

$$(\omega^{n-1}(t))':=\frac{\partial}{\partial t}(\omega_{i_1j_1}(t)\ldots\omega_{i_{n-1}j_{n-1}}(t))\,dz^{i_1}\wedge d\overline{z}^{j_1}\wedge\ldots dz^{i_{n-1}}\wedge d\overline{z}^{j_{n-1}}\in\mathcal{A}^{n-1,n-1}(M).$$

Then, if every metric ω_t is balanced, for $t\in I$, it must hold that

$$\partial\circ i_{\varphi'(0)}(\omega^{n-1})=-\overline{\partial}(\omega^{n-1}(0))'.$$

Given Theorem 2.4.1, it is straightforward to see that the following cohomological obstruction holds.

Corollary 2.4.2 ([130])**.** *Let (M,J) be a compact Hermitian manifold endowed with a balanced metric g and associated fundamental form ω. If there exists a smooth family of balanced metrics which coincides with ω in $t=0$, along the family of deformations $\{M_t\}_t$ with $M_0=M$ and parametrized by the $(0,1)$-vector form $\varphi(t)$ on M, then the following equation must hold*

$$\left[\partial\circ i_{\varphi'(0)}(\omega^{n-1})\right]_{H^{n-1,n}_{\overline{\partial}}(M)}=0.$$

Proof (of Theorem 2.4.1). The metrics ω_t are balanced for every $t\in I$, i.e., $\overline{\partial}_t\omega_t^{n-1}=0$. By means of the extension map, this equation can be written as

$$\overline{\partial}_t\left(e^{i_\varphi|i_{\overline{\varphi}}}(\omega^{n-1}(t))\right)=0,\qquad\forall t\in I.$$

Also, formula (2.1.3) implies that $\overline{\partial}_t\omega_t^{n-1} = 0$ for every $t \in I$ if and only if

$$e^{i_\varphi|i_{\overline{\varphi}}}\Big((I-\overline{\varphi}\varphi)^{-1}\lrcorner([\partial, i_\varphi] + \overline{\partial})(I-\overline{\varphi}\varphi)\lrcorner(\omega^{n-1}(t))\Big) = 0, \quad \forall t \in I.$$

We now use equation (2.1.1) and we expand in Taylor series centered in $t = 0$ the term $\overline{\partial}_t\omega_t^{n-1}$, noting that

$$\varphi = \varphi(t) = t\varphi'(0) + o(t)$$

and, therefore,

$$(I-\varphi\overline{\varphi}) = (I-\overline{\varphi}\varphi) = (I-\varphi\overline{\varphi})^{-1} = (I-\overline{\varphi}\varphi)^{-1} = I + o(t),$$

to obtain

$$\begin{aligned}\overline{\partial}_t\omega_t^{n-1} &= \Big(I + t\varphi'(0)\lrcorner + t\overline{\varphi'(0)}\lrcorner\Big)\lrcorner\Big(([\partial, t\varphi'(0)\lrcorner] + \overline{\partial})(\omega^{n-1}(0) + t(\omega^{n-1}(0))')\Big) + o(t)\\ &= \Big(I + t\varphi'(0)\lrcorner + t\overline{\varphi'(0)}\lrcorner\Big)\lrcorner\Big(t\partial(\varphi'(0)\lrcorner\omega^{n-1}(0)) + t\overline{\partial}(\omega^{n-1}(0))')\Big) + o(t)\\ &= t\partial(\varphi'(0)\lrcorner\omega^{n-1}(0)) + t\overline{\partial}(\omega^{n-1}(0))') + o(t)\end{aligned}$$

Since $\overline{\partial}_t\omega_t^{n-1} = 0$ holds true for every $t \in I$, if we differentiate it with respect to t in $t = 0$, we obtain

$$\frac{\partial}{\partial t}\Big|_{t=0}\left(\overline{\partial}_t\omega_t^{n-1}\right) = \frac{\partial}{\partial t}\Big|_{t=0}\left[t\partial(\varphi'(0)\lrcorner\omega^{n-1}(0)) + t\overline{\partial}(\omega^{n-1}(0))') + o(t)\right] = 0.$$

Hence,

$$\partial(\varphi'(0)\lrcorner\omega^{n-1}) + \overline{\partial}(\omega^{n-1}(0))' = 0,$$

therefore concluding the proof. □

We now apply Theorem 2.4.1 and Corollary 2.4.2 to find obstructions on each family of non-Kähler complex parallelisable solvmanifolds as characterized in [105]. In particular, we will focus on the complex parallelisable Nakamura manifold and the Iwasawa manifold.

2.4.1 Example 1

(Complex parallelisable Nakamura manifold). Let $G := \mathbb{C} \ltimes_\gamma \mathbb{C}^2$ be the complex Lie group given by the action of $\mathbb{C}$ on $\mathbb{C}^2$, via

$$\gamma(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z}\end{pmatrix}.$$

Let us consider the discrete subgroup Γ of G of the form $\Gamma := (\mathbb{Z}(a+ib) + \mathbb{Z}(c+id)) \ltimes_\gamma \Gamma''$, where

- the set Γ'' is a lattice of $\mathbb{C}^2$;
- the complex numbers $a + ib$ and $c + id$ are such that $\mathbb{Z}(a+ib) + \mathbb{Z}(c+id)$ is a lattice in $\mathbb{C}$;
- the matrices $\gamma(a+ib)$ and $\gamma(c+id)$ are conjugates in $SL(4;\mathbb{Z})$, where we regard $SL(2;\mathbb{C}) \subset SL(4;\mathbb{R})$.

Then Γ is a lattice of G and the compact quotient $M := \Gamma/G$ is called the *complex parallelisable Nakamura Manifold*, see [105, Section 2] for details on its construction.

It is well known that G is a solvable non nilpotent Lie group, therefore the quotient M is a 3-dimensional solvmanifold, which is biholomorphic to $\mathbb{C}^3$.

If $\{z^1\}$ and $\{z^2, z^3\}$ are the standard coordinates on respectively $\mathbb{C}$ and $\mathbb{C}^2$, a left-invariant frame of $(1,0)$-vector fields on G is given by $\{Z_1, Z_2, Z_3\}$, where

$$\begin{cases} Z_1 = \frac{\partial}{\partial z^1} \\ Z_2 = e^{z^1}\frac{\partial}{\partial z^1} \\ Z_3 = e^{-z^1}\frac{\partial}{\partial z^3} \end{cases}$$

and the dual coframe of $(1,0)$-differential forms in $\mathcal{A}^{1,0}(M)$ is given by $\{\eta^1, \eta^2, \eta^3\}$, where

$$\begin{cases} \eta^1 = dz^1 \\ \eta^2 = e^{-z^1}dz^2 \\ \eta^3 = e^{z^1}dz^3. \end{cases}$$

Note that structure equations

$$\begin{cases} d\eta^1 = 0 \\ d\eta^2 = -\eta^1 \wedge \eta^2 \\ d\eta^3 = \eta^1 \wedge \eta^3. \end{cases} \tag{2.4.2}$$

imply that the coframe of left-invariant $(1,0)$-forms $\{\eta^1, \eta^2, \eta^3\}$ induce an almost complex left-invariant structure J on M, which is integrable.

From now on, we adopt the abbreviation for the wedge product of differential forms, i.e., for example, $\eta^{ij\bar{k}} := \eta^i \wedge \eta^j \wedge \bar{\eta}^k$.

Let us consider a generic left-invariant Hermitian metric g on (M, J), with associated fundamental form ω given by

$$\omega = \frac{i}{2}\sum_{j=1}^{3}\alpha_{j\bar{j}}\eta^{j\bar{j}} + \frac{1}{2}\sum_{j<k}(\alpha_{j\bar{k}} - \bar{\alpha}_{j\bar{k}})\eta^{j\bar{k}},$$

with coefficients $\alpha_{j\bar{k}} \in \mathbb{C}$, for $j, k \in \{1,2,3\}$, such that the matrix representing g

$$\begin{pmatrix} \alpha_{1\bar{1}} & -i\alpha_{1\bar{2}} & -i\alpha_{1\bar{3}} \\ i\bar{\alpha}_{1\bar{2}} & \alpha_{2\bar{2}} & -i\alpha_{2\bar{3}} \\ i\bar{\alpha}_{1\bar{3}} & i\bar{\alpha}_{2\bar{3}} & \alpha_{3\bar{3}} \end{pmatrix}$$

is positive definite. From structure equations (2.4.2), it is easy to check that $\bar{\partial}\omega^2 = 0$, hence any left-invariant Hermitian metric on (M, J) is balanced.

We notice that the dimension of the space $H^{0,1}_{\bar{\partial}}(M)$ depends on the choice of the lattice $\Gamma = (\mathbb{Z}(a+ib) + \mathbb{Z}(c+id)) \ltimes_\gamma \Gamma''$, in particular on the choice of the real numbers b and d. More accurately, it can be proved that, if $b, d \in 2\pi\mathbb{Z}$, then $\dim H^{0,1}_{\bar{\partial}}(M) = 3$, whereas, if either $b \notin 2\pi\mathbb{Z}$ or $d \notin 2\pi\mathbb{Z}$, then $\dim H^{0,1}_{\bar{\partial}}(M) = 1$, see [81]. Hence, we distinguish two cases.

Case (i): $b, d \in 2\pi\mathbb{Z}$

We define the following $\mathbb{C}$-base for $\mathcal{A}^{0,1}(M)$, consisting of the left-invariant $(0,1)$-forms $\{\tilde{\eta}^1, \tilde{\eta}^2, \tilde{\eta}^3\}$, defined as

$$\begin{cases} \tilde{\eta}^1 := \bar{\eta}^1 \\ \tilde{\eta}^2 := e^{\bar{z}^1 - z^1}\bar{\eta}^2 \\ \tilde{\eta}^3 := e^{z^1 - \bar{z}^1}\bar{\eta}^3, \end{cases}$$

where the functions $e^{\bar{z}^1-z^1}$ and $e^{z^1-\bar{z}^1}$ are well defined on M because of the choice of the lattice Γ.

Accordingly to [105, Section 3], small deformations of (M,J) can be characterized by means of the $(0,1)$-vector form

$$\varphi(\mathbf{t}) = \sum_{i,j=1}^{3} t_{ij}\tilde{\eta}^j \otimes Z_i,$$

with the coefficients of $\mathbf{t} = (t_{11}, t_{12}, t_{13}, t_{21}, t_{22}, t_{23}, t_{31}, t_{32}, t_{33}) \in B(0,\delta) \subset \mathbb{C}^9$, $\delta > 0$, belonging to one of the following classes:

$$t_{11} \neq 0, t_{12} = t_{13} = t_{23} = t_{32} = 0; \tag{2.4.3}$$

$$t_{11} = t_{22} = t_{33} = 0; \tag{2.4.4}$$

$$t_{12} \neq 0, t_{11} = t_{13} = t_{21} = t_{23} = t_{31} = 0; \tag{2.4.5}$$

$$t_{13} \neq 0, t_{11} = t_{12} = t_{21} = t_{31} = t_{32} = 0. \tag{2.4.6}$$

We can now make use of Theorem 2.4.1 and Corollary 2.4.2 to find obstruction for each class of small deformations of (M,J).

Class (2.4.3). In this case, the $(0,1)$-vector form parametrizing the deformation is

$$\varphi(\mathbf{t}) = t_{11}\tilde{\eta}^1 \otimes Z_1 + t_{21}\tilde{\eta}^1 \otimes Z_2 + t_{22}\tilde{\eta}^2 \otimes Z_2 + t_{31}\tilde{\eta}^1 \otimes Z_3 + t_{33}\tilde{\eta}^3 \otimes Z_3,$$

for $\mathbf{t} = (t_{11}, t_{21}, t_{22}, t_{31}, t_{33}) \in B(0,\delta) \subset \mathbb{C}^5$, $\delta > 0$. We then consider the smooth curve of deformations

$$t \mapsto \varphi(t) := t\,(a_{11}\tilde{\eta}^1 \otimes Z_1 + a_{21}\tilde{\eta}^1 \otimes Z_2 + a_{22}\tilde{\eta}^2 \otimes Z_2 + a_{31}\tilde{\eta}^1 \otimes Z_3 + a_{33}\tilde{\eta}^3 \otimes Z_3) \in \mathcal{A}^{0,1}(T^{1,0}(M))$$

for $t \in I = (-\epsilon,\epsilon)$, $\epsilon > 0$, $(a_{11}, a_{21}, a_{22}, a_{31}, a_{33}) \in \mathbb{C}^5$, whose derivative in $t = 0$ is

$$\varphi'(0) = a_{11}\tilde{\eta}^1 \otimes Z_1 + a_{21}\tilde{\eta}^1 \otimes Z_2 + a_{22}\tilde{\eta}^2 \otimes Z_2 + a_{31}\tilde{\eta}^1 \otimes Z_3 + a_{33}\tilde{\eta}^3 \otimes Z_3.$$

With the aid of (2.4.2), we compute

$$\begin{aligned}\partial \circ i_{\varphi'(0)}(\omega^2) &= [a_{12}(i\alpha_{1\bar{1}}\alpha_{2\bar{3}} + \overline{\alpha}_{1\bar{2}}\alpha_{1\bar{3}}) + a_{32}(i\alpha_{3\bar{3}}\overline{\alpha}_{1\bar{2}} - \overline{\alpha}_{1\bar{3}}\alpha_{2\bar{3}})]e^{\bar{z}^1-z^1}\eta^{12} \wedge \eta^{\bar{1}\bar{2}\bar{3}} \\ &+ \frac{1}{2}\left[a_{11}(i\alpha_{2\bar{2}}\alpha_{1\bar{3}} - \alpha_{1\bar{2}}\alpha_{2\bar{3}}) + a_{31}(|\alpha_{2\bar{3}}|^2 - \alpha_{2\bar{2}}\alpha_{3\bar{3}})\right]\eta^{12} \wedge \eta^{\bar{1}\bar{2}\bar{3}} \\ &+ [a_{13}(\alpha_{1\bar{2}}\overline{\alpha}_{1\bar{3}} - i\alpha_{1\bar{3}}\overline{\alpha}_{2\bar{3}}) + a_{23}(i\alpha_{2\bar{2}}\overline{\alpha}_{1\bar{3}} + \overline{\alpha}_{1\bar{2}}\overline{\alpha}_{2\bar{3}})]e^{z^1-\bar{z}^1}\eta^{13} \wedge \eta^{\bar{1}\bar{2}\bar{3}} \\ &+ \frac{1}{2}[a_{11}(i\alpha_{1\bar{1}}\alpha_{1\bar{2}} + \alpha_{1\bar{3}}\overline{\alpha}_{2\bar{3}}) + a_{21}(|\alpha_{2\bar{3}}|^2 - \alpha_{2\bar{2}}\alpha_{3\bar{3}})]\eta^{13} \wedge \eta^{\bar{1}\bar{2}\bar{3}}.\end{aligned}$$

We note that the forms $e^{\bar{z}^1-z^1}\eta^{12} \wedge \eta^{\bar{1}\bar{2}\bar{3}}$ and $e^{z^1-\bar{z}^1}\eta^{13} \wedge \eta^{\bar{1}\bar{2}\bar{3}}$ are $\overline{\partial}$-exact. In fact,

$$\begin{aligned}e^{\bar{z}^1-z^1}\eta^{12} \wedge \eta^{\bar{1}\bar{2}\bar{3}} &= \overline{\partial}(e^{\bar{z}^1-z^1}\eta^{12} \wedge \tilde{\eta}^{23}) \\ e^{z^1-\bar{z}^1}\eta^{13} \wedge \eta^{\bar{1}\bar{2}\bar{3}} &= \overline{\partial}(-e^{z^1-\bar{z}^1}\eta^{13} \wedge \tilde{\eta}^{23}),\end{aligned}$$

therefore they both represent a vanishing class in $H^{2,3}_{\overline{\partial}}(M)$. On the other hand, it can be easily shown that the forms $\eta^{12} \wedge \eta^{\bar{1}\bar{2}\bar{3}}$ and $\eta^{13} \wedge \eta^{\bar{1}\bar{2}\bar{3}}$ are harmonic with respect to the Dolbeault Laplacian operator, i.e., they belong to $\mathcal{H}^{2,3}_{\overline{\partial}}(M,g)$. As a consequence, they correspond, respectively, to non-vanishing cohomology classes $[\eta^{12} \wedge \eta^{\bar{1}\bar{2}\bar{3}}]_{\overline{\partial}}$ and $[\eta^{13} \wedge \eta^{\bar{1}\bar{2}\bar{3}}]_{\overline{\partial}}$ in $H^{2,3}_{\overline{\partial}}(X)$. Hence, by Corollary 2.4.2, if one of the following equations does not hold

$$\begin{cases} a_{11}(i\alpha_{1\bar{1}}\alpha_{1\bar{2}} + \alpha_{1\bar{3}}\overline{\alpha}_{2\bar{3}}) + a_{21}(|\alpha_{2\bar{3}}|^2 - \alpha_{2\bar{2}}\alpha_{3\bar{3}}) = 0 \\ a_{11}(i\alpha_{2\bar{2}}\alpha_{1\bar{3}} - \alpha_{1\bar{2}}\alpha_{2\bar{3}}) + a_{31}(|\alpha_{2\bar{3}}|^2 - \alpha_{2\bar{2}}\alpha_{3\bar{3}}) = 0, \end{cases}$$

there exists no curve of balanced metrics $\{\omega_t\}_{t\in I}$ such that $\omega_0=\omega$ along the curve of deformations $t\mapsto\varphi(t)$.

Class (2.4.4). The deformation is parametrized by the $(0,1)$-vector form $\varphi(\mathbf{t})$, with

$$\varphi(\mathbf{t})=t_{12}\tilde{\eta}^2\otimes Z_1+t_{13}\tilde{\eta}^3\otimes Z_1+t_{21}\tilde{\eta}^1\otimes Z_2+t_{23}\tilde{\eta}^3\otimes Z_2+t_{31}\tilde{\eta}^1\otimes Z_3+t_{32}\tilde{\eta}^2\otimes Z_3,$$

with $\mathbf{t}=(t_{12},t_{13},t_{21},t_{23},t_{31},t_{32})\in B(0,\delta)\subset\mathbb{C}^6$, $\delta>0$.

We consider the smooth curve of deformations

$$\begin{aligned}t\mapsto\varphi(t):=&t(a_{12}\tilde{\eta}^2\otimes Z_1+a_{13}\tilde{\eta}^3\otimes Z_1+a_{21}\tilde{\eta}^1\otimes Z_2\\&+a_{23}\tilde{\eta}^3\otimes Z_2+a_{31}\tilde{\eta}^1\otimes Z_3+a_{32}\tilde{\eta}^2\otimes Z_3),\end{aligned}$$

for $t\in I=(-\epsilon,\epsilon),\epsilon>0$, whose derivative in $t=0$ is

$$\begin{aligned}\varphi'(0)=&a_{12}\tilde{\eta}^2\otimes Z_1+a_{13}\tilde{\eta}^3\otimes Z_1+a_{21}\tilde{\eta}^1\otimes Z_2\\&+a_{23}\tilde{\eta}^3\otimes Z_2+a_{31}\tilde{\eta}^1\otimes Z_3+a_{32}\tilde{\eta}^2\otimes Z_3.\end{aligned}$$

With the aid of (2.4.2), we compute

$$\begin{aligned}\partial\circ i_{\varphi'(0)}(\omega^2)=&[a_{12}(i\alpha_{1\bar{1}}\alpha_{2\bar{3}}+\overline{\alpha}_{1\bar{2}}\alpha_{1\bar{3}})+a_{32}(i\alpha_{3\bar{3}}\overline{\alpha}_{1\bar{2}}-\overline{\alpha}_{1\bar{3}}\alpha_{2\bar{3}})]e^{\bar{z}^1-z^1}\eta^{12}\wedge\eta^{\tilde{1}\tilde{2}\tilde{3}}\\&+\frac{1}{2}\left[a_{31}(|\alpha_{2\bar{3}}|^2-\alpha_{2\bar{2}}\alpha_{3\bar{3}})\right]\eta^{12}\wedge\eta^{\tilde{1}\tilde{2}\tilde{3}}\\&+[a_{13}(\alpha_{1\bar{2}}\overline{\alpha}_{1\bar{3}}-i\alpha_{1\bar{3}}\overline{\alpha}_{2\bar{3}})+a_{23}(i\alpha_{2\bar{2}}\overline{\alpha}_{1\bar{3}}+\overline{\alpha}_{1\bar{2}}\overline{\alpha}_{2\bar{3}})]e^{z^1-\bar{z}^1}\eta^{13}\wedge\eta^{\tilde{1}\tilde{2}\tilde{3}}\\&+\frac{1}{2}[a_{21}(|\alpha_{2\bar{3}}|^2-\alpha_{2\bar{2}}\alpha_{3\bar{3}})]\eta^{13}\wedge\eta^{\tilde{1}\tilde{2}\tilde{3}}.\end{aligned}$$

We observe that, again, since the forms $e^{\bar{z}^1-z^1}\eta^{12}\wedge\eta^{\tilde{1}\tilde{2}\tilde{3}}$ and $e^{z^1-\bar{z}^1}\eta^{13}\wedge\eta^{\tilde{1}\tilde{2}\tilde{3}}$ are cohomologous to 0 in $H^{2,3}_{\overline{\partial}}(M)$ and the forms $\eta^{12}\wedge\eta^{\tilde{1}\tilde{2}\tilde{3}}$ and $\eta^{13}\wedge\eta^{\tilde{1}\tilde{2}\tilde{3}}$ are $\overline{\partial}$-harmonic, the obstruction from Corollary 2.4.2 boils down to

$$\begin{aligned}a_{21}(|\alpha_{2\bar{3}}|^2-\alpha_{2\bar{2}}\alpha_{3\bar{3}})=0\\a_{31}(|\alpha_{2\bar{3}}|^2-\alpha_{2\bar{2}}\alpha_{3\bar{3}})=0.\end{aligned}$$

We point out that, since the metric g is Hermitian and, hence, positive definite, the real number $|\alpha_{2\bar{3}}|^2-\alpha_{2\bar{2}}\alpha_{3\bar{3}}$ is strictly positive. Therefore, there exists no curve of balanced metrics $\{\omega_t\}_{t\in I}$ such that $\omega_0=\omega$ along the curve of deformations $t\mapsto\varphi(t)$, if

$$\begin{pmatrix}a_{21}\\a_{31}\end{pmatrix}\neq\begin{pmatrix}0\\0\end{pmatrix}.$$

Class (2.4.5). For this class, the $(0,1)$-vector deformation form is

$$\varphi(\mathbf{t})=t_{12}\tilde{\eta}^2\otimes Z_1+t_{22}\tilde{\eta}^2\otimes Z_2+t_{32}\tilde{\eta}^2\otimes Z_3+t_{33}\tilde{\eta}^3\otimes Z_3,$$

for $\mathbf{t}=(t_{12},t_{22},t_{32},t_{33})\in B(0,\delta)\subset\mathbb{C}^4$, $\delta>0$. We consider the smooth curve of deformations

$$t\mapsto\varphi(t):=t(a_{12}\tilde{\eta}^2\otimes Z_1+a_{22}\tilde{\eta}^2\otimes Z_2+a_{32}\tilde{\eta}^2\otimes Z_3+a_{33}\tilde{\eta}^3\otimes Z_3),$$

for $t\in I=(-\epsilon,\epsilon),\epsilon>0$, whose derivative in $t=0$ is

$$\varphi'(0)=a_{12}\tilde{\eta}^2\otimes Z_1+a_{22}\tilde{\eta}^2\otimes Z_2+a_{32}\tilde{\eta}^2\otimes Z_3+a_{33}\tilde{\eta}^3\otimes Z_3.$$

In this case, $\partial \circ i_{\varphi'(0)}(\omega^2) = 0$, therefore Corollary 2.4.2 gives no obstruction to the existence of smooth curves of balanced metrics $\{\omega_t\}_{t\in I}$ such that $\omega_0 = \omega$ along the curve of deformations $t \mapsto \varphi(t)$. Moreover, if $\{\omega_t\}_{t\in I}$ is any smooth curve of left-invariant Hermitian metrics along $\varphi(t)$ such that $\omega_0 = \omega$, we can see that $\overline{\partial}(\omega^2(0))' = 0$, where we have set $\omega_t = e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}\omega(t)$, for $\omega(t) = \omega_{ij}(t)dz^i \wedge d\bar{z}^j \in \mathcal{A}^{1,1}(M)$. Therefore, also Theorem 2.4.1 yields no obstruction.

Class (2.4.6). The $(0,1)$-vector form for this class is

$$\varphi(\mathbf{t}) = t_{13}\tilde{\eta}^3 \otimes Z_1 + t_{22}\tilde{\eta}^2 \otimes Z_2 + t_{23}\tilde{\eta}^3 \otimes Z_2 + t_{33}\tilde{\eta}^3 \otimes Z_3,$$

for $\mathbf{t} = (t_{13}, t_{22}, t_{23}, t_{33}) \in B(0,\delta) \subset \mathbb{C}^4$, $\delta > 0$.

Let us consider the smooth curve of deformations

$$t \mapsto \varphi(t) := t(a_{13}\tilde{\eta}^3 \otimes Z_1 + a_{22}\tilde{\eta}^2 \otimes Z_2 + a_{23}\tilde{\eta}^3 \otimes Z_2 + a_{33}\tilde{\eta}^3 \otimes Z_3)$$

for $t \in (-\epsilon, \epsilon)$ and its derivative in $t = 0$

$$\varphi'(0) = a_{13}\tilde{\eta}^3 \otimes Z_1 + a_{22}\tilde{\eta}^2 \otimes Z_2 + a_{23}\tilde{\eta}^3 \otimes Z_2 + a_{33}\tilde{\eta}^3 \otimes Z_3.$$

Also in this case, $\partial \circ i_{\varphi'(0)}(\omega^2) = 0$, i.e., Corollary 2.4.2 yields no obstruction and analogously to the previous class, also Theorem 2.4.1 yields no non-trivial conditions.

We can focus now on the other case.

Case (ii): $c \notin 2\pi\mathbb{Z}$ **or** $d \notin 2\pi\mathbb{Z}$

In [105, Section 3], it is shown that $H^{0,1}_{\overline{\partial}}(M) = \mathbb{C}\langle\overline{\eta}^1\rangle$, and any small deformation of (M,J) can be parametrized by the $(0,1)$-vector form

$$\varphi(\mathbf{t}) := t_1\overline{\eta}^1 \otimes Z_1 + t_2\overline{\eta}^1 \otimes Z_2 + t_3\overline{\eta}^1 \otimes Z_3,$$

with $\mathbf{t} = (t_1, t_2, t_3) \in B(0,\delta) \subset \mathbb{C}^3$, $\delta > 0$. We can then consider the smooth curve of deformations

$$t \mapsto \varphi(t) := t(a_1\overline{\eta}^1 \otimes Z_1 + a_2\overline{\eta}^1 \otimes Z_2 + a_3\overline{\eta}^1 \otimes Z_3),$$

for $t \in (-\epsilon, \epsilon)$, $\epsilon > 0$, $(a_1, a_2, a_3) \in \mathbb{C}^3$, whose derivative in $t = 0$ is

$$\varphi'(0) = a_1\overline{\eta}^1 \otimes Z_1 + a_2\overline{\eta}^1 \otimes Z_2 + a_3\overline{\eta}^1 \otimes Z_3.$$

By making use of (2.4.2), we compute

$$\begin{aligned}\partial \circ i_{\varphi'(0)}(\omega^2) = &\frac{1}{2}(a_2(|\alpha_{2\bar{3}}|^2 - \alpha_{2\bar{2}}\alpha_{3\bar{3}}) + a_1(i\alpha_{3\bar{3}}\alpha_{1\bar{2}} + \alpha_{1\bar{3}}\overline{\alpha}_{2\bar{3}}))\eta^{13\overline{123}}\\ &+\frac{1}{2}(a_3(|\alpha_{2\bar{3}}|^2 - \alpha_{2\bar{2}}\alpha_{3\bar{3}}) + a_1(i\alpha_{2\bar{2}}\alpha_{1\bar{3}} - \alpha_{1\bar{2}}\alpha_{2\bar{3}}))\eta^{12\overline{123}}.\end{aligned}$$

We can easily verify that $\overline{\partial}\eta^{12\overline{123}} = \overline{\partial}^*\eta^{12\overline{123}} = \overline{\partial}\eta^{13\overline{123}} = \overline{\partial}^*\eta^{13\overline{123}} = 0$, i.e., the $(2,3)$-forms $\eta^{12\overline{123}}$ and $\eta^{13\overline{123}}$ are $\overline{\partial}$–harmonic. Therefore, the Dolbeault cohomology classes $[\eta^{12\overline{123}}]_{H^{2,3}_{\overline{\partial}}(M)}$ and $[\eta^{13\overline{123}}]_{H^{2,3}_{\overline{\partial}}(M)}$ are not vanishing. On this accounts, Corollary 2.4.2 implies that if there exists a smooth curve of balanced metrics $\{\omega_t\}_{t\in I}$ along the smooth curve of deformations $t \mapsto \varphi(t)$, then we must have that

$$\begin{cases} a_2(|\alpha_{2\bar{3}}|^2 - \alpha_{2\bar{2}}\alpha_{3\bar{3}}) + a_1(i\alpha_{3\bar{3}}\alpha_{1\bar{2}} + \alpha_{1\bar{3}}\overline{\alpha}_{2\bar{3}}) = 0\\ a_3(|\alpha_{2\bar{3}}|^2 - \alpha_{2\bar{2}}\alpha_{3\bar{3}}) + a_1(i\alpha_{2\bar{2}}\alpha_{1\bar{3}} - \alpha_{1\bar{2}}\alpha_{2\bar{3}}) = 0.\end{cases} \tag{2.4.7}$$

We notice that, if $a_1 = 0$, i.e., $\varphi'(0) = a_2\overline{\eta}^1 \otimes Z_2 + a_3\overline{\eta}^1 \otimes Z_3$, condition (2.4.7) becomes

$$\begin{cases} a_2 = 0 \\ a_3 = 0 \end{cases}$$

since $|\alpha_{2\overline{3}}|^2 - \alpha_{2\overline{2}}\alpha_{3\overline{3}} \neq 0$, being g a Hermitian metric. Hence, by Corollary 2.4.2, we can conclude that there exists no smooth curve of balanced metrics $\{\omega_t\}_{t\in I}$ such that $\omega_0 = \omega$, along $\varphi(t)$ with $\varphi'(0) = a_2\overline{\eta}^1 \otimes Z_2 + a_3\overline{\eta}^1 \otimes Z_3$.

Viceversa, let us consider the case in which $a_1 \neq 0$ and at least one between a_2 and a_3 vanishes, i.e., for example, $a_2 = 0$. Then, condition (2.4.7) reduces to

$$\begin{cases} a_1 = 0 \\ a_3(|\alpha_{2\overline{3}}|^2 - \alpha_{2\overline{2}}\alpha_{3\overline{3}}) + a_1(i\alpha_{2\overline{2}}\alpha_{1\overline{3}} - \alpha_{1\overline{2}}\alpha_{2\overline{3}}) = 0, \end{cases}$$

since the term $i\alpha_{2\overline{2}}\alpha_{1\overline{3}} - \alpha_{1\overline{2}}\alpha_{2\overline{3}} \neq 0$, being g a Hermitian metric. We assumed $a_1 \neq 0$, therefore by Corollary 2.4.2, there exists no smooth curve of balanced metrics $\{\omega_t\}_{t\in I}$ such that $\omega_0 = \omega$, along the smooth curve of deformations $\varphi(t)$ with $\varphi'(0) = a_1\overline{\eta}^1 \otimes Z_1 + a_3\overline{\eta}^1 \otimes Z_3$. We come to the same conclusion if we consider $a_3 = 0$.

We can then summarize what we obtained in the following theorems.

Theorem 2.4.3 ([130]). *Let (M, J) be the complex parallelisable Nakamura manifold with $\dim H^{0,1}_{\overline{\partial}}(M) = 3$, where J is the integrable left-invariant almost complex structure induced by the left-invariant coframe $\{\eta^1, \eta^2, \eta^3\}$ with structure equations*

$$\begin{cases} d\eta^1 = 0 \\ d\eta^2 = -\eta^{12} \\ d\eta^3 = \eta^{13}. \end{cases}$$

Let g be any left-invariant Hermitian (balanced) metric with associated fundamental form

$$\omega = \frac{i}{2}\sum_{j=1}^{3} \alpha_{j\overline{j}}\eta^{j\overline{j}} + \frac{1}{2}\sum_{j<k}(\alpha_{j\overline{k}} - \overline{\alpha}_{j\overline{k}})\eta^{j\overline{k}}.$$

Defining the left-invariant $(0,1)$-forms $\{\tilde{\eta}^1, \tilde{\eta}^2, \tilde{\eta}^3\}$ by

$$\begin{aligned} \tilde{\eta}^1 &:= \overline{\eta}^1 \\ \tilde{\eta}^2 &:= e^{\overline{z}^1 - z^1}\overline{\eta}^2 \\ \tilde{\eta}^3 &:= e^{z^1 - \overline{z}^1}\overline{\eta}^3, \end{aligned}$$

let $t \mapsto \varphi(t) := t\sum_{i,j=1}^{3} a_{ij}\tilde{\eta}^j \otimes Z_i \in \mathcal{A}^{0,1}(T^{1,0}(M))$ be a smooth curve of deformations of (M, J), for $\{a_{ij}\}_{i,j=1}^{3} \subset \mathbb{C}$, $t \in I = (-\epsilon, \epsilon)$, $\epsilon > 0$.

Then,

- *if $a_{11} \neq 0, a_{12} = a_{13} = a_{23} = a_{32} = 0$, there exists no smooth curve of balanced metrics $\{\omega_t\}_{t\in I}$ such that $\omega_0 = \omega$, along the curve of deformation $t \mapsto \varphi(t)$, if*

$$\begin{pmatrix} a_{11}(i\alpha_{2\overline{2}}\alpha_{1\overline{3}} - \alpha_{1\overline{2}}\alpha_{2\overline{3}}) + a_{31}(|\alpha_{2\overline{3}}|^2 - \alpha_{2\overline{2}}\alpha_{3\overline{3}}) = 0 \\ a_{11}(i\alpha_{1\overline{1}}\alpha_{1\overline{2}} + \alpha_{1\overline{3}}\overline{\alpha}_{2\overline{3}}) + a_{21}(|\alpha_{2\overline{3}}|^2 - \alpha_{2\overline{2}}\alpha_{3\overline{3}}) = 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

- *if* $a_{11} = a_{22} = a_{33} = 0$, *there exists no smooth curve of balanced metrics* $\{\omega_t\}_{t\in I}$ *such that* $\omega_0 = \omega$, *along the curve of deformation* $t \mapsto \varphi(t)$, *if*

$$\begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Theorem 2.4.4 ([130])**.** *Let* (M, J) *be the complex parallelisable Nakamura manifold with* $\dim H^{0,1}_{\partial}(M) = 1$, *where* J *is the integrable left-invariant almost complex structure induced by the left-invariant coframe* $\{\eta^1, \eta^2, \eta^3\}$ *with structure equations*

$$\begin{cases} d\eta^1 = 0 \\ d\eta^2 = -\eta^{12} \\ d\eta^3 = \eta^{13}. \end{cases}$$

Let g *be any left-invariant Hermitian (balanced) metric with associated fundamental form*

$$\omega = \frac{i}{2}\sum_{j=1}^{3} \alpha_{j\bar{j}}\eta^{j\bar{j}} + \frac{1}{2}\sum_{j<k}(\alpha_{j\bar{k}} - \overline{\alpha}_{j\bar{k}})\eta^{j\bar{k}}.$$

Let $t \mapsto \varphi(t) := t\sum_i^3 a_i \overline{\eta}^1 \otimes Z_i \in \mathcal{A}^{0,1}(T^{1,0}(M))$ *be a smooth curve of deformations of* (M, J), *for* $0 \neq (a_1, a_2, a_3) \in \mathbb{C}^3$, $t \in I = (-\epsilon, \epsilon)$, $\epsilon > 0$.

Then, there exists no smooth curve of balanced metrics $\{\omega_t\}_{t\in I}$ *such that* $\omega_0 = \omega$, *along the curve of deformation* $t \mapsto \varphi(t)$, *if*

$$\begin{pmatrix} a_2(|\alpha_{2\bar{3}}|^2 - \alpha_{2\bar{2}}\alpha_{3\bar{3}}) + a_1(i\alpha_{3\bar{3}}\alpha_{1\bar{2}} + \alpha_{1\bar{3}}\overline{\alpha}_{2\bar{3}}) \\ a_3(|\alpha_{2\bar{3}}|^2 - \alpha_{2\bar{2}}\alpha_{3\bar{3}}) + a_1(i\alpha_{2\bar{2}}\alpha_{1\bar{3}} - \alpha_{1\bar{2}}\alpha_{2\bar{3}}) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In particular, if one the following holds:

- $a_1 = 0$;
- $a_1 \neq 0$, $(a_2, a_3) \in \{(a_2, 0), (0, a_3)\}$,

there exists no smooth curve of balanced metrics $\{\omega_t\}_{t\in I}$ *such that* $\omega_0 = \omega$, *along the curve of deformation* $t \mapsto \varphi(t)$.

2.4.2 Example 2

(Iwasawa manifold). Let $G = H(3; \mathbb{C})$ be the 3-dimensional complex Heisenberg group. It well known that G is a 2-step nilpotent Lie group. Let us consider the lattice $\Gamma := H(3, \mathbb{Z}[i])$ of G, i.e., $\Gamma = H(3;\mathbb{C}) \cap \mathrm{GL}(3; \mathbb{Z}[i])$. The quotient $M := \Gamma/G$ is a compact manifold, known as the *Iwasawa manifold*. In particular, M is a 3-dimensional 2-step complex nilmanifold with universal covering $\mathbb{C}^3$.

If $\{z^1, z^2, z^3\}$ are the standard coordinates on $\mathbb{C}^3$, the forms $\{\eta^1, \eta^2, \eta^3\}$, defined by

$$\begin{cases} \eta^1 = dz^1 \\ \eta^2 = dz^2 \\ \eta^3 = dz^3 - z^1 dz^2, \end{cases}$$

are a left-invariant coframe of $(1,0)$-forms on G, therefore they descend to the quotient M. The dual frame of $(1,0)$-vector fields $\{Z_1, Z_2, Z_3\}$ on G has local expression

$$\begin{cases} Z_1 = \frac{\partial}{\partial z^1} + z^1 \frac{\partial}{\partial z^3} \\ Z_2 = \frac{\partial}{\partial z^2} \\ Z_3 = \frac{\partial}{\partial z^3}. \end{cases}$$

We notice that, by looking at structure equations

$$\begin{cases} d\eta^1 = 0 \\ d\eta^2 = 0 \\ d\eta^3 = -\eta^{12}, \end{cases} \tag{2.4.8}$$

the coframe $\{\eta^1, \eta^2, \eta^3\}$ induces a left-invariant almost complex structure J on M, which is integrable.

Let g be any left-invariant Hermitian metric on (M, J). Its associated fundamental form ω can be written as

$$\omega = \frac{i}{2} \sum_{j=1}^{3} \alpha_{j\bar{j}} \eta^{j\bar{j}} + \frac{1}{2} \sum_{j<k} (\alpha_{j\bar{k}} - \overline{\alpha}_{j\bar{k}}) \eta^{j\bar{k}},$$

with complex numbers $\{\alpha_{j\bar{k}}\}_{j,k=1}^{3}$ such that the matrix respresenting g

$$\begin{pmatrix} \alpha_{1\bar{1}} & -i\alpha_{1\bar{2}} & -i\alpha_{1\bar{3}} \\ i\overline{\alpha}_{1\bar{2}} & \alpha_{2\bar{2}} & -i\alpha_{2\bar{3}} \\ i\overline{\alpha}_{1\bar{3}} & i\overline{\alpha}_{2\bar{3}} & \alpha_{3\bar{3}} \end{pmatrix}$$

is positive definite. By structure equations (2.4.8), it is easy to check that $\overline{\partial}\omega^2 = 0$, i.e., the left-invariant Hermitian metric g is balanced.

In [105], Nakamura gives a complete description of Kuranishi space of the Iwasawa manifold. In particular, any small deformation of (M, J) can be parametrized by the $(0,1)$-vector form

$$\varphi(\mathbf{t}) = \sum_{i=1}^{3} \sum_{j=1}^{2} t_{ij} \overline{\eta}^j \otimes Z_i - (t_{11}t_{22} - t_{12}t_{21}) \overline{\eta}^3 \otimes Z_3,$$

with $\mathbf{t} = (t_{11}, t_{12}, t_{21}, t_{22}, t_{31}, t_{32}) \in B(0, \delta) \subset \mathbb{C}^6$, $\delta > 0$.

Let us consider the smooth curve of deformations

$$\begin{aligned} t \mapsto \varphi(t) :=& t(a_{11}\overline{\eta}^1 \otimes Z_1 + a_{12}\overline{\eta}^2 \otimes Z_1 + a_{21}\overline{\eta}^1 \otimes Z_2 + a_{22}\overline{\eta}^2 \otimes Z_2 + a_{31}\overline{\eta}^1 \otimes Z_3 \\ &+ a_{32}\overline{\eta}^2 \otimes Z_3) - t^2(a_{11}a_{22} - a_{12}a_{21})\overline{\eta}^3 \otimes Z_3 \in \mathcal{A}^{0,1}(T^{1,0}(M)), \end{aligned}$$

with $t \in I = (-\epsilon, \epsilon)$, $\epsilon > 0$ and $(a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}) \in \mathbb{C}^6$. Its derivative in $t = 0$ is

$$\varphi'(0) = a_{11}\overline{\eta}^1 \otimes Z_1 + a_{12}\overline{\eta}^2 \otimes Z_1 + a_{21}\overline{\eta}^1 \otimes Z_2 + a_{22}\overline{\eta}^2 \otimes Z_2 + a_{31}\overline{\eta}^1 \otimes Z_3 a_{32}\overline{\eta}^2 \otimes Z_3.$$

With the aid of structure equations (2.4.8), we compute

$$\begin{aligned} \partial \circ i_{\varphi'(0)}(\omega^2) =& \frac{1}{2}\Big(a_{12}(|\alpha_{1\bar{3}}|^2 - \alpha_{1\bar{1}}\alpha_{3\bar{3}}) + a_{21}(\alpha_{2\bar{2}}\alpha_{3\bar{3}} - |\alpha_{2\bar{3}}|^2) \\ & - a_{11}(i\alpha_{3\bar{3}}\alpha_{1\bar{2}} + \alpha_{1\bar{3}}\overline{\alpha}_{2\bar{3}}) + a_{22}(-i\alpha_{3\bar{3}}\overline{\alpha}_{1\bar{3}} + \overline{\alpha}_{1\bar{3}}\alpha_{2\bar{3}}) \Big) \eta^{12\bar{1}\bar{2}\bar{3}}. \end{aligned}$$

We notice that the $(2,3)$-form $\eta^{12\overline{123}}$ is both $\overline{\partial}$-closed and $\overline{\partial}^*$-closed, i.e., it is $\overline{\partial}$-harmonic. Hence, the corresponding Dolbeault class $[\eta^{12\overline{123}}]_{\overline{\partial}}$ is non-vanishing in $H^{2,3}_{\overline{\partial}}(M)$. Applying Corollary 2.4.2, we see that there exists no smooth curve of balanced metrics $\{\omega_t\}_{t\in I}$ along the curve of deformations $t \mapsto \varphi(t)$, such that $\omega_0 = \omega$, if the following equation holds

$$a_{12}(|\alpha_{1\bar{3}}|^2 - \alpha_{1\bar{1}}\alpha_{3\bar{3}}) + a_{21}(\alpha_{2\bar{2}}\alpha_{3\bar{3}} - |\alpha_{2\bar{3}}|^2) - a_{11}(i\alpha_{3\bar{3}}\alpha_{1\bar{2}} + \alpha_{1\bar{3}}\overline{\alpha}_{2\bar{3}}) + a_{22}(-i\alpha_{3\bar{3}}\overline{\alpha}_{1\bar{3}} + \overline{\alpha}_{1\bar{3}}\alpha_{2\bar{3}}) \neq 0.$$

We observe that, for $a_{ij} = 0$ for $(i,j) \neq (1,2)$, we find the same curve of deformations that Alessandrini and Bassanelli costructed in [8] to prove the non stability of the balanced condition under small deformations of the complex structure.

We gather what we have obtained in the following theorem.

Theorem 2.4.5 ([130]). *Let (M,J) be the Iwasawa manifold with integrable left-invariant complex structure J, induced by the left-invariant coframe $\{\eta^1,\eta^2,\eta^3\}$ with structure equations*

$$\begin{cases} d\eta^1 = 0 \\ d\eta^2 = 0 \\ d\eta^3 = -\eta^{12}. \end{cases}$$

Let g be a left-invariant Hermitian (balanced) metric on (M,J) with associated fundamental form

$$\omega = \frac{i}{2}\sum_{j=1}^{3}\alpha_{j\bar{j}}\eta^{j\bar{j}} + \frac{1}{2}\sum_{j<k}(\alpha_{j\bar{k}} - \overline{\alpha}_{j\bar{k}})\eta^{j\bar{k}}.$$

Let $t \mapsto \varphi(t) = t(\sum_{i=1}^{3}\sum_{j=1}^{2} a_{ij}\overline{\eta}^j \otimes Z_j) - t^2(a_{11}a_{22} - a_{12}a_{21})\overline{\eta}^3 \otimes Z_3 \in \mathcal{A}^{0,1}(T^{1,0}(M))$ be a smooth curve of deformations of (M,J), with $\{a_{ij}\}_{i=1}^{3}{}_{j=1}^{2} \subset \mathbb{C}$, $t \in I = (-\epsilon,\epsilon)$, $\epsilon > 0$.

Then, if the following condition holds

$$a_{12}(|\alpha_{1\bar{3}}|^2 - \alpha_{1\bar{1}}\alpha_{3\bar{3}}) + a_{21}(\alpha_{2\bar{2}}\alpha_{3\bar{3}} - |\alpha_{2\bar{3}}|^2) - a_{11}(i\alpha_{3\bar{3}}\alpha_{1\bar{2}} + \alpha_{1\bar{3}}\overline{\alpha}_{2\bar{3}}) + a_{22}(-i\alpha_{3\bar{3}}\overline{\alpha}_{1\bar{3}} + \overline{\alpha}_{1\bar{3}}\alpha_{2\bar{3}}) \neq 0,$$

there exists no smooth curve of balanced metrics $\{\omega_t\}_{t\in I}$ such that $\omega_0 = \omega$ along the curve of deformations $t \mapsto \varphi(t)$.

Chapter 3

p-Kähler and balanced structures on nilmanifolds with nilpotent complex structures

In this chapter, we will first determine obtructions to the existence of p-Kähler forms, as recalled in section 1.5, on nilmanifolds endowed with a invariant nilpotent complex structure. In particular, we will determine an optimal p such that there exist non p-Kähler structures on such complex manifolds. Then, we will study in detail the existence of special structures on the Bigalke-Rollenske manifolds M^{4n-2} (such a family of $4n-2$ dimensional complex non-Kähler manifolds, $n \geq 2$, were introduced in [25] to show that the degeneration step of the Frölicher spectral sequence can be arbitrarily high). Using the mentioned obstructions, we will show that the Bigalke-Rollenske manifolds do not admit any p-Kähler form, $p \in \{1, \dots, 4n-4\}$, except for $p = 4n-3$, i.e., they admit a balanced metric, thus proving that, unlike the Kähler setting, on a balanced manifold the degeneracy step of the Frölicher spectral sequence can be arbitrarily high, adding to the results in [119], where it was shown that the existence of a balanced metric does not imply the degeneration at the first step. In fact, whereas the degeneracy step of the Frölicher spectral sequence on a non-Kähler manifold might be higher than one, as first shown in [84] (see also [43]), weaker metric conditions might impose restrictions on the degeneration of the Frölicher spectral sequence. Note that, starting from Bigalke-Rollenske manifolds, Kasuya and Stelzig in [82] have recently constructed compact complex manifolds which provide counterexamples to Popovici's conjecture [120, Conjecture 1.3] on the relation between the existence of SKT metrics and degeneration of the Frölicher spectral sequence at the second page on a compact non-Kähler manifold.

3.1 p-Kähler structures on nilmanifolds with nilpotent complex structures

We begin this section by recalling the following lemma by Hind, Medori, and Tomassini, which provides a geneal obstruction to existence of p-Kähler structures on complex mnaifold, see [71, Proposition 3.4].

Lemma 3.1.1 ([71]). *Let (M, J) be a compact complex manifold of complex dimension n. Suppose that there exists a non-closed $(2n-2p-1)$-form η such that the $(n-p, n-p)$-component of $d\eta$ satisfies*

$$(d\eta)^{n-p,n-p} = \sum_k c_k \psi_k \wedge \bar{\psi}_k$$

where the ψ_k are simple $(n-p,0)$-forms and the c_k have the same sign. Then, (M,J) does not admit a p-Kähler form.

We will use this lemma to prove the non-existence of a p-Kähler form (for suitable p) on nilmanifolds with nilpotent complex structures.

Let $M = \Gamma\backslash G$ be a nilmanifold of complex dimension n and let J be an invariant complex structure on M, i.e., J is induced by a left-invariant complex structure on G. We denote with $\mathfrak{g}$ the Lie algebra of G. Recall that J is a *nilpotent* complex structure if, and only if, there exists a co-frame of invariant $(1,0)$-forms $\{\eta^i\}_{i=1}^n$ satisfying

$$d\eta^k \in \mathrm{Span}_{\mathbb{C}}\langle \eta^{ij}, \eta^{i\bar{j}}\rangle_{i,j=1,\dots,n-1}, \qquad k = 1,\cdots,n.$$

Then, we prove the following.

Theorem 3.1.2 ([131]). *Let $M = \Gamma\backslash G$ be a nilmanifold of complex dimension n endowed with a invariant nilpotent complex structure J. With the above notations, let k be the index such that*

$$d\eta^i = 0 \quad \textit{for} \quad i = 1,\cdots,k \quad \textit{and} \quad d\eta^i \neq 0 \quad \textit{for} \quad i = k+1,\cdots,n. \tag{3.1.1}$$

Then, there are no $(n-k)$-Kähler forms on M.

Proof. In order to prove the result we will exhibit a $(2k-1)$-form α satisfying the hypothesis of Lemma 3.1.1.

Since $d\eta^{k+1} \neq 0$ then at least one between $\partial\eta^{k+1}$ and $\bar{\partial}\eta^{k+1}$ is different from 0. Suppose now that $\bar{\partial}\eta^{k+1} \neq 0$. We will deal later with the other case. Since J is nilpotent,

$$\bar{\partial}\eta^{k+1} = \sum_{l,m=1}^{k} C_{l\bar{m}}\,\eta^{l\bar{m}} \neq 0$$

for some constants $C_{l\bar{m}}$. Hence, we fix two indices $i,j \leq k$ such that $C_{i\bar{j}} \neq 0$.
We define the following $(2k-1)$-form

$$\alpha = \eta^{1\cdots\hat{i}\cdots k+1\,\bar{1}\cdots\hat{\bar{j}}\cdots\bar{k}},$$

where $\hat{\eta}^i$ and $\hat{\bar{\eta}}^j$ mean that we are removing the forms η^i and $\bar{\eta}^j$ from α.
By the structure equations, since $d\eta^i = 0$ for $i = 1,\cdots,k$ and J is nilpotent,

$$d\alpha = \pm C_{i\bar{j}}\,\eta^{1\cdots k\,\bar{1}\cdots\bar{k}}$$

hence α satisfies the hypothesis of Lemma 3.1.1 and so there is no $(n-k)$-Kähler structure on M.
On the other side, suppose that $\bar{\partial}\eta^{k+1} = 0$ and $\partial\eta^{k+1} \neq 0$.
Since J is nilpotent,

$$\partial\eta^{k+1} = \sum_{l,m=1,l<m}^{k} A_{l\,m}\,\eta^{l\,m} \neq 0$$

for some constants $A_{l\,m}$. Hence, we fix two indices $i < j \leq k$ such that $A_{i\,j} \neq 0$.
We define the following $(2k-1)$-form

$$\alpha = \eta^{1\cdots\hat{i}\cdots\hat{j}\cdots k+1\,\bar{1}\cdots\bar{k}}.$$

By the structure equations, since $d\eta^i = 0$ for $i = 1,\cdots,k$ and J is nilpotent,

$$d\alpha = \pm A_{i\,j}\eta^{1\cdots k\,\bar{1}\cdots\bar{k}}$$

hence α satisfies the hypothesis of Lemma 3.1.1 and so there is no $(n-k)$-Kähler structure on M. ∎

As a Corollary for $k = 1$ one gets immediately

Corollary 3.1.3 ([131]). *Let $M = \Gamma\backslash G$ be a nilmanifold of complex dimension n endowed with an invariant nilpotent complex structure J, with co-frame of $(1,0)$-forms $\{\eta^i\}_{i=1,\cdots,n}$ satisfying the following structure equations,*

$$d\eta^1 = 0 \quad \text{and} \quad d\eta^i \neq 0 \quad \text{for} \quad i = 2,\cdots,n.$$

Then, there are no balanced metrics on M.

We notice that there are large classes of complex nilmanifolds where Theorem 3.1.2 can be applied. For instance, if J is *abelian*, namely $[Jx, Jy] = [x, y]$ for every $x, y \in \mathfrak{g}$, or *bi-invariant*, namely $J[x, y] = [Jx, y]$ for every $x, y \in \mathfrak{g}$, then it is nilpotent (cf. [126]). Moreover, by [110] if (M, J) is a 2-step nilmanifold with invariant complex structure and J-invariant center, then J is nilpotent.

Remark 3.1.4. We notice that the nilpotency of the complex structure of the nilmanifold is crucial in the previous results. Indeed, when the hypotesis of nilpotency on the complex structure of the nilmanifold is dropped, Theorem 3.1.2 and Corollary 3.1.3 are not valid in general. More precisely, in [40] the authors consider the real 6-dimensional nilmanifold, whose associated Lie algebra is $\mathfrak{h}_{19}^{-} = (0, 0, 0, 12, 23, 14-35)$ and they prove that it is endowed with invariant non nilpotent complex structures (see [40, Theorem 2.1]) which satisfy condition (3.1.1) for $k = 1$ ([40, Table 2]), indeed the complex structure equations are

$$d\eta^1 = 0, \quad d\eta^2 = \eta^{13} + \eta^{1\bar{3}}, \quad d\eta^3 = \pm i\left(\eta^{1\bar{2}} - \eta^{2\bar{1}}\right).$$

As shown in [40, Remark 5.4] such nilmanifolds admit invariant balanced metrics, i.e., 2-Kähler forms.

We now show that $p = n-k$ in Theorem 3.1.2 is optimal. Indeed, we will show now two examples of 2-step nilmanifolds with invariant abelian complex structures that admit a $(n-k-1)$-Kähler form and a $(n-k+1)$-Kähler form.

Example 3.1.5. Let M be the 2-step nilmanifold of complex dimension 3 with abelian complex structure defined by the following structure equations

$$d\eta^1 = d\eta^2 = 0, \qquad d\eta^3 = \eta^{1\bar{2}}$$

where $\{\eta^i\}_{i=1,2,3}$ is a co-frame of $(1,0)$-forms.
With the previous notations we have $n = 3$ and $k = 2$. So, by Theorem 3.1.2 there are no 1-Kähler forms on M. Of course, this was already known since on non-toral nilmanifolds there are no Kähler metrics.
Now, we show that there exists a 2-Kähler form on M, namely a $(n-k+1)$-Kähler form.
Let

$$\Omega := -\eta^{1\bar{1}2\bar{2}} - \eta^{1\bar{1}3\bar{3}} - \eta^{2\bar{2}3\bar{3}}.$$

Then, Ω is a real transverse $(2,2)$-form and by the structure equations

$$d\Omega = 0\,.$$

Hence, Ω is a 2-Kähler form on M. In particular, there exists a balanced metric ω on M such that $\omega^2 = \Omega$. In fact, it is easy to see that

$$\omega = i\eta^{1\bar{1}} + i\eta^{2\bar{2}} + i\eta^{3\bar{3}}.$$

Example 3.1.6. Let M be the 8-dimensional 2-step nilmanifold with abelian complex structure defined by the following structure equations

$$d\eta^1 = 0, \qquad d\eta^2 = d\eta^3 = d\eta^4 = \eta^{1\bar{1}}$$

where $\{\eta^i\}_{i=1,2,3,4}$ is a co-frame of $(1,0)$-forms.
With the previous notations we have $n = 4$ and $k = 1$. So, by Corollary 3.1.3 there are no balanced metrics on M.
Now, we show that there exists a 2-Kähler form on M, namely a $(n-k-1)$-Kähler form.
Let

$$\begin{aligned}\Omega := -\eta^{1\bar{1}2\bar{2}} - \eta^{1\bar{1}3\bar{3}} - \eta^{1\bar{1}4\bar{4}} - \eta^{2\bar{2}3\bar{3}} - \eta^{2\bar{2}4\bar{4}} - \eta^{3\bar{3}4\bar{4}} + \\ + \eta^{2\bar{2}3\bar{4}} + \eta^{2\bar{2}4\bar{3}} + \eta^{2\bar{4}3\bar{3}} + \eta^{4\bar{2}3\bar{3}} + \eta^{2\bar{3}4\bar{4}} + \eta^{3\bar{2}4\bar{4}}\,.\end{aligned}$$

Then, Ω is a real transverse $(2,2)$-form and by the structure equations one can see directly that

$$d\Omega = 0\,.$$

Hence, Ω is a 2-Kähler form on M.

3.2 Special Hermitian metrics on the Bigalke and Rollenske's manifolds

In this section, we discuss the existence of special Hermitian metrics and p-Kähler forms on the 2-step nilmanifolds with the nilpotent complex structure constructed by Bigalke and Rollenske in [25]. In particular, for every $n \geq 2$, these $(4n-2)$-dimensional compact complex manifolds are such that the Frölicher spectral sequence does not degenerate at the E_n term.

We start by recalling the construction. Fix $n \geq 2$ and let G_n be the real nilpotent subgroup of $GL(2n+2,\mathbb{C})$ consisting of the matrices of the form

$$\begin{pmatrix}
1 & 0 & & & & & \cdots & & 0 & \bar{y}_1 & w_1 \\
 & 1 & 0 & \cdots & 0 & \bar{z}_1 & -x_1 & 0 & \cdots & 0 & w_2 \\
 & & \ddots & & & & \ddots & & & \vdots & \vdots \\
 & & & 1 & 0 & \cdots & 0 & \bar{z}_{n-1} & -x_{n-1} & 0 & w_n \\
 & & & & 1 & 0 & & \cdots & & 0 & y_1 \\
 & & & & & \ddots & & & & \vdots & \vdots \\
 & & & & & & & & & & \\
 & & & & & & & \ddots & & \vdots & \vdots \\
 & & & & & & & & 1 & 0 & y_n \\
 & & & & & & & & & 1 & z_1 \\
 & & & & & & & & & & 1
\end{pmatrix}.$$

with $x_1,\ldots,x_{n-1},y_1,\ldots,y_n,z_1,\ldots,z_{n-1},w_1,\ldots,w_n \in \mathbb{C}$.

Let Γ be the subgroup of G_n consisting of the matrices of the same form and entries in $\mathbb{Z}[i]$. Then, Γ is a discrete uniform subgroup of G_n and the quotient $M^{4n-2} := \Gamma\backslash G_n$ is a compact $(4n-2)$-dimensional 2-step nilmanifold with an invariant complex structure. A global co-frame of invariant $(1,0)$-forms is given by

$$dx_1,\ldots,dx_{n-1},dy_1,\ldots,dy_n,dz_1,\ldots,dz_{n-1},\omega_1,\ldots,\omega_n$$

where

$$\omega_1 = dw_1 - \bar{y}_1 dz_1, \quad \omega_k = dw_k - \bar{z}_{k-1}dy_{k-1} + x_{k-1}dy_k \quad (k = 2,\ldots,n).$$

The structure equations become

$$d(dx_j) = d(dz_j) = 0, \qquad (j = 1, \dots, n-1)$$

$$d(dy_j) = 0, \qquad (j = 1, \dots, n)$$

$$\partial\omega_1 = 0, \quad \overline{\partial}\omega_1 = dz_1 \wedge d\bar{y}_1$$

$$\partial\omega_j = dx_{j-1} \wedge dy_j, \quad \overline{\partial}\omega_j = dy_{j-1} \wedge d\bar{z}_{j-1} \qquad (j = 2, \dots, n)$$

In [25] the authors show that that the Frölicher spectral sequence of M^{4n-2} has non-vanishing differential d_n, namely the Frölicher spectral sequence does not degenerate at the E_n term.

We now rename the forms dx_j, dy_j, dz_j, and ω_j by considering the basis of $(1,0)$-forms $\{\eta^j\}_{j=1}^{4n-2}$, defined as follows

$$\eta^j := \begin{cases} dx_j, & 1 \le j < n \\ dy_j, & n \le j < 2n \\ dz_j, & 2n \le j < 3n-1 \\ \omega_j, & 3n-1 \le j \le 4n-2. \end{cases}$$

As a result, the structure equations become

$$d\eta^j = \begin{cases} 0, & 1 \le j < 3n-1 \\ \eta^{2n} \wedge \overline{\eta}^n, & j = 3n-1 \\ \eta^{j-3n+1} \wedge \eta^{j-2n+1} + \eta^{j-2n} \wedge \overline{\eta}^{j-n}, & 3n \le j \le 4n-2, \end{cases} \tag{3.2.1}$$

or, more precisely,

$$\partial\eta^j = \begin{cases} 0, & 1 \le j \le 3n-1 \\ \eta^{j-3n+1} \wedge \eta^{j-2n+1}, & 3n \le j \le 4n-2, \end{cases} \tag{3.2.2}$$

and

$$\overline{\partial}\eta^j = \begin{cases} 0, & 1 \le j < 3n-1 \\ \eta^{2n} \wedge \overline{\eta}^n, & j = 3n-1 \\ \eta^{j-2n} \wedge \overline{\eta}^{j-n}, & 3n \le j \le 4n-2. \end{cases} \tag{3.2.3}$$

Now we study the existence of special Hermitian metrics on Bigalke and Rollenske's nilmanifolds. In particular, one can apply Theorem 3.1.2 and get immediately the following proposition.

Proposition 3.2.1 ([131]). *For every $n \ge 2$ the Bigalke and Rollenske's nilmanifold M^{4n-2} does not admit any n-Kähler form.*

In fact, we can show more, namely there are no p-Kähler forms except for balanced metrics. More precisely, we prove the following theorem.

Theorem 3.2.2 ([131]). *For every $n \ge 2$ the Bigalke and Rollenske's nilmanifold M^{4n-2} does not admit any p-Kähler form for $1 \le p < 4n-3$.*

Proof. We will show that on any Bigalke and Rollenske's manifold M^{4n-2}, for every fixed p, with $1 \le p < 4n-3$, we can construct a non closed $(8n-2p-5)$-form α such that the $(4n-2-p, 4n-2-p)$-component of $d\alpha_p$ satisfies

$$(d\alpha_p)^{(4n-2-p,4n-2-p)} = \epsilon_p \psi_p \wedge \overline{\psi_p}, \tag{3.2.4}$$

with $\psi_p \in \mathcal{A}^{4n-2-p,0}(M^{4n-2})$ a simple form and $\epsilon_p \in \{-1,1\}$. By Lemma 3.1.1, this will assure that there exists no p-Kähler form on M^{4n-2}.

Let us consider separately the cases
(i) $1 \leq p < n$;
(ii) $n \leq p \leq 4n-2$.
Before doing so, we remark that, by structure equations (3.2.1), the index j such that every term of the expression of $d\eta^j$ contains forms with the highest indices, is $j = 4n-2$. Such expression is

$$d\eta^{4n-2} = \eta^{n-1} \wedge \eta^{2n-1} + \eta^{2n-2} \wedge \overline{\eta}^{3n-2},$$

whereas, in general, we have that $d\eta^j$, $d\overline{\eta}^j \neq 0$ if, and only if, $3n-1 \leq j \leq 4n-2$.

(i) Even though it is well-known that on non-toral nilmanifolds there are no 1-Kahler forms, since they coincide with Kähle metrics, we will consider the case $p = 1$ for the benefit of the following constructions. We must construct a non closed $(8n-7)$-form satisfying property (3.2.4). In particular, if we start from the $(8n-4)$-form

$$\eta^1 \wedge \cdots \wedge \eta^{4n-2} \wedge \overline{\eta}^1 \wedge \cdots \wedge \overline{\eta}^{4n-2},$$

we must remove three 1-forms. For this purpose, we select η^{2n-2}, $\overline{\eta}^{3n-2}$, and $\overline{\eta}^{4n-2}$, therefore considering the $(4n-3, 4n-4)$-form α_1 given by

$$\alpha_1 = \eta^1 \wedge \cdots \wedge \eta^{2\hat{n}-2} \wedge \cdots \wedge \eta^{4n-2} \wedge \overline{\eta}^1 \wedge \cdots \wedge \overline{\eta}^{3\hat{n}-2} \wedge \cdots \wedge \overline{\eta}^{4n-3}.$$

We now compute the $(4n-3, 4n-3)$-component of $d\alpha_1$. By the structure equations (3.2.3), we remark that the only non trivial relevant differentials are

$$\begin{aligned}\overline{\partial}\eta^{3n-1} &= \eta^{2n} \wedge \overline{\eta}^n,\\ \overline{\partial}\eta^j &= \eta^{j-2n} \wedge \overline{\eta}^{j-n}, \quad 3n \leq j \leq 4n-2.\end{aligned}$$

In order to have a non vanishing term, we must ensure that $\overline{\partial}\eta^j = \eta^{2n-2} \wedge \overline{\eta}^{3n-2}$. However, this can happen if and only if $j = 4n-2$, resulting in

$$\begin{aligned}d\alpha_1^{(4n-3,4n-3)} &= d\left(\eta^1 \wedge \cdots \wedge \eta^{2\hat{n}-2} \wedge \cdots \wedge \eta^{4n-2} \wedge \overline{\eta}^1 \wedge \cdots \wedge \overline{\eta}^{3\hat{n}-2} \wedge \cdots \wedge \overline{\eta}^{4n-3}\right)\\ &= \eta^1 \wedge \cdots \wedge \eta^{4n-3} \wedge \overline{\eta}^1 \wedge \cdots \wedge \overline{\eta}^{4n-3}.\end{aligned}$$

Thus, considering $\psi_1 := \eta^1 \wedge \cdots \wedge \eta^{4n-3} \in \mathcal{A}^{4n-3,0}(M^{4n-2})$, we can conclude by Lemma 3.1.1.

Therefore, for the case $1 < p < n$, we can construct α_p starting from the $(8n-7)$-form α_1 and then remove the forms $\eta^{3n-1}, \eta^{3n}, \ldots \eta^{3n+p-3}, \overline{\eta}^{3n-1}, \overline{\eta}^{3n}, \ldots, \overline{\eta}^{3n+p-3}$, (which accounts to removing $2p-2$ forms), obtaining a non closed $(8n-2p-5)$-form. Then, the $(4n-2-p, 4n-2-p)$-component of $d\alpha_p$ is of type

$$\psi_p \wedge \overline{\psi}_p,$$

with $\psi_p \in \mathcal{A}^{4n-2-p,0}(M^{4n-2})$ given by

$$\psi_p = \eta^1 \wedge \cdots \wedge \eta^{3n-2} \wedge \eta^{3n+p-2} \wedge \cdots \wedge \eta^{4n-3}.$$

Again, we can conclude by Lemma 3.1.1.
(ii) Let us now consider the case $n \leq p \leq 4n-2$, starting from $p = n$ for the benefit of the following construction.
We must find a $(6n-5)$-form α_n such that the $(3n-2, 3n-2)$-component of $d\alpha_n$ satisfies condition (3.2.4). We construct the form α_n as we have previously done, setting

$$\alpha_n = \eta^1 \wedge \cdots \wedge \eta^{2\hat{n}-2} \wedge \cdots \wedge \eta^{3n-2} \wedge \eta^{4n-2} \wedge \overline{\eta}^1 \wedge \cdots \wedge \overline{\eta}^{3n-3},$$

with $\alpha_n \in \mathcal{A}^{3n-2,3n-3}(M^{4n-2})$. By structure equations, we see that we have removed all the forms with non trivial differential but $d\eta^{4n-2}$. Therefore, when computing the differential $d\alpha_n$, we obtain

$$d\alpha_n = -\eta^1 \wedge \cdots \wedge \eta^{3n-2} \wedge \overline{\eta}^1 \wedge \cdots \wedge \overline{\eta}^{3n-2}.$$

By setting $\psi_n := \eta^1 \wedge \cdots \wedge \eta^{3n-2}$, we conclude by Lemma 3.1.1.
Now, if $n+1 \leq p < 4n-3$, we construct the $(8n-2p-5)$-form α_p starting from the $(6n-5)$-form α_n and then removing the forms $\eta^1, \ldots, \eta^{p-n}, \overline{\eta}^1, \ldots, \overline{\eta}^{p-n}$. We clarify that, for $p-n \geq 2n-2$, since η^{2n-2} has already been removed, we keep removing the $(1,0)$-forms with higher index starting from η^{2n-1}, whereas we keep $\overline{\eta}^{2n-2}$ and remove $\overline{\eta}^{2n-1}$ and so forth, so to remove $(1,0)$-forms for a total of $p-n$ forms and $(0,1)$-forms for a total of $p-n$ forms. This procedure accounts to building $\alpha_p \in \mathcal{A}^{4n-p-2,4n-p-3}$ as

$$\alpha_p = \eta^{p-n+1} \wedge \cdots \wedge \widehat{\eta^{2n-2}} \wedge \cdots \wedge \eta^{3n-2} \wedge \eta^{4n-2} \wedge \overline{\eta}^{p-n+1} \wedge \cdots \wedge \overline{\eta}^{3n-3}$$

if $n+1 \leq p < 3n-3$, and

$$\alpha_p = \eta^{p-n+2} \wedge \cdots \wedge \eta^{3n-2} \wedge \eta^{4n-2} \wedge \overline{\eta}^{2n-2} \wedge \overline{\eta}^{p-n+1} \wedge \cdots \wedge \overline{\eta}^{3n-3},$$

if $3n-3 \leq p \leq 4n-4$. We then compute $d\alpha_p$. Since the only non trivial differential is $d\eta^{4n-2}$, we obtain

$$d\alpha_p = \epsilon_p \eta^{p-n+1} \wedge \cdots \wedge \cdots \wedge \eta^{3n-2} \wedge \overline{\eta}^{p-n+2} \wedge \cdots \wedge \overline{\eta}^{3n-2}$$

if $n+1 \leq p < 3n-3$, and

$$d\alpha_p = \epsilon_p \eta^{2n-2} \wedge \eta^{p-n+1} \wedge \cdots \wedge \eta^{3n-2} \wedge \overline{\eta}^{2n-2} \wedge \overline{\eta}^{p-n+2} \wedge \cdots \wedge \overline{\eta}^{3n-3},$$

if $3n-3 \leq p \leq 4n-4$. The number $\epsilon_p \in \{\pm 1\}$ is a sign term. Therefore, by setting

$$\psi_p = \eta^{p-n+1} \wedge \cdots \wedge \cdots \wedge \eta^{3n-2}$$

for $n+1 \leq p < 3n-3$ and

$$\psi_p = \eta^{2n-2} \wedge \eta^{p-n+1} \wedge \cdots \wedge \eta^{3n-2},$$

if $3n-3 \leq p \leq 4n-4$, we can finally conclude by Lemma 3.1.1. □

However, we show that there exist $(4n-3)$-Kähler forms. More precisely, we prove the following theorem.

Theorem 3.2.3 ([131])**.** *For every $n \geq 2$ the Bigalke and Rollenske's nilmanifold M^{4n-2} admits balanced metrics.*

Proof. We show that the diagonal Hermitian metric

$$\omega := \frac{i}{2} \sum_{j=1}^{4n-2} \eta^j \wedge \overline{\eta}^j$$

is balanced, i.e., $d\omega^{4n-3} = 0$. Notice that

$$\omega^{4n-3} = \left(\frac{i}{2}\right)^{4n-3} \frac{1}{(4n-3)!} \sum_{k=1}^{4n-2} \eta^1 \wedge \overline{\eta}^1 \wedge \cdots \wedge \hat{\eta}^k \wedge \hat{\overline{\eta}}^k \wedge \cdots \wedge \eta^{4n-2} \wedge \overline{\eta}^{4n-2}.$$

We denote by $\alpha_k := \eta^1 \wedge \overline{\eta}^1 \wedge \cdots \wedge \hat{\eta}^k \wedge \hat{\overline{\eta}}^k \wedge \cdots \wedge \eta^{4n-2} \wedge \overline{\eta}^{4n-2}$. From the structure equations, when we compute $d\omega^{4n-3}$ we consider separately each term

$$d\alpha_k = d(\eta^1 \wedge \overline{\eta}^1 \wedge \cdots \wedge \hat{\eta}^k \wedge \hat{\overline{\eta}}^k \wedge \cdots \wedge \eta^{4n-2} \wedge \overline{\eta}^{4n-2}).$$

By the structure equations we have that $d\alpha_k = 0$ for every $k = 1, \dots, 4n-2$. Indeed, by Leibniz rule, the only way to have $d\alpha^k \neq 0$ would be that for some index $j = 1, \dots, \hat{k}, \dots, 4n-2$, $d\eta^j$ or $d\bar{\eta}^j$ contains exactly $\eta^k \wedge \bar{\eta}^k$. But this is not the case as showed by the structure equations. Hence, $d\alpha_k = 0$ for every $k = 1, \dots, 4n-2$ and so $d\omega^{4n-3} = 0$ and so ω is balanced.

□

As a consequence, combining this with [25, Theorem 1], we get that there is no relation between the existence of balanced metrics and the degeneracy step of the Frölicher spectral sequence.

Corollary 3.2.4 ([131]). *On balanced manifolds the degeneracy step of the Frölicher spectral sequence can be arbitrarily large.*

In particular, this is in contrast with the situation in Kähler geometry where for compact Kähler manifolds the Frölicher spectral sequence degenerates at the first step and with a conjecture by Popovici stating that on compact SKT manifolds the Frölicher spectral sequence degenerates at the second step (cf. [120, Conjecture 1.3]). In fact, in relation with this conjecture we show explicitly the following.

Proposition 3.2.5 ([131]). *For every $n \geq 2$ the Bigalke and Rollenske's nilmanifold M^{4n-2} does not admit any SKT metric.*

Proof. In order to show that M^{4n-2} does not admit any SKT metric we use the characterization of [48] in terms of currents. More precisely, we will construct a non-zero positive $(1,1)$-current which is $\partial\bar{\partial}$-exact. Indeed, by a direct computation using the structure equations

$$\psi := \eta^1 \wedge \bar{\eta}^1 \wedge \dots \wedge \eta^{4n-3} \wedge \bar{\eta}^{4n-3} =$$

$$\partial\bar{\partial}\left(\eta^1 \wedge \bar{\eta}^1 \wedge \dots \wedge \hat{\eta}^{n-1} \wedge \hat{\bar{\eta}}^{n-1} \wedge \dots \wedge \hat{\eta}^{2n-1} \wedge \hat{\bar{\eta}}^{2n-1} \wedge \dots \wedge \eta^{4n-2} \wedge \bar{\eta}^{4n-2}\right).$$

The $(4n-3, 4n-3)$-form ψ gives rise to a $\partial\bar{\partial}$-exact non-zero positive $(1,1)$-current on M. □

Notice that this follows also by [56] where the authors show that on non-tori nilmanifolds balanced and SKT metrics cannot coexist. We recall that an Hermitian metric ω on a complex manifold is called *locally conformally Kähler* if

$$d\omega = \theta \wedge \omega$$

where θ is a d-closed 1-form. We then have immediately the following proposition.

Proposition 3.2.6 ([131]). *For every $n \geq 2$ the Bigalke and Rollenske's nilmanifold M^{4n-2} does not admit any locally conformally Kähler metric.*

Proof. This follows directly combining Theorem 3.2.3 and [110, Theorem 4.9] where it is proved that on non-tori complex nilmanifolds endowed with an invariant complex structure, locally conformally Kähler metrics and balanced metrics cannot coexist. □

Chapter 4

Dolbeault and Bott-Chern formalities: deformations and $\partial\overline{\partial}$-lemma

In this chapter we study the behaviour of the complex formalities for a complex manifold recalled in section 1.4 under the action of deformations of the complex structure. In particular, completing the picture started with the non-openness theorems for Dolbeaul formality by Tomassini and Torelli (see [150]), respectively, Tardini and Tomassini (see [147]), we prove that the properties of being Dolbeault formal, admitting a Dolbeault formal metric, and the vanishing of every Dolbeault Massey products are not closed properties under deformations, see Theorem 4.2.1. Analogously, the property of admitting a geometrically Bott-Chern formal metric and the vanishing of every Aeppli-Bott-Chern Massey product are not closed under deformations of the complex structure, see Theorem 4.3.1. In particular, we prove Theorem 4.2.1 by constructing a holomorphic family of compact complex manifolds $\{M_t\}_{t\in D}$ obtained as a deformation of the complex structure of the holomorphically parallelizable Nakamura manifold, such that each M_t is geometrically Dolbeault formal and Dolbeault formal for $t \in D \setminus \{0\}$, but M_0 has a non vanishing Dolbeault-Massey triple product. This will assure that on each M_t every triple Dolbeault-Massey product is vanishing, for $t \in D \setminus \{0\}$, but M_0 is neither Dolbeault formal, nor geometrically Dolbeault formal. To prove Theorem 4.3.1, we use a different presentation of the holomorphically parallelizable Nakamura manifold selecting a suitable family of lattices. Then, we consider a holomorphic deformation of the complex structure such that each M_t is geometrically-Bott-Chern-formal, for $t \neq 0$, but M_0 has a non vanishing ABC-Massey product, hence on each M_t every ABC-Massey product vanishes but M_0 is not geometrically-Bott-Chern-formal.

We then further investigate the notion of Aeppli-Bott-Chern Massey triple product and we highlight an interesting behaviour, i.e., we are able to provide an example of a smooth non-Kähler complex manifold which satisfies the $\partial\overline{\partial}$-lemma but admits a non vanishing Aeppli-Bott-Chern-Massey product, see Theorem 4.4.1. This is in contrast with the Sullivan (respectively, Dolbeault) formality setting, since a manifold satisying the $\partial\overline{\partial}$-lemma is both Sullivan formal and Dolbeault formal, hence every Massey (respectively, Dolbeault Massey) product vanishes on such a manifold. In order to prove Theorem 4.4.1, we start by constructing a complex orbifold obtained as a quotient of the Iwasawa manifold and by showing that it satisfies the $\partial\overline{\partial}$-lemma and it admits a non vanishing ABC-Massey product. Then, we explicitly construct a smooth resolution $\tilde{M}$ of such orbifold and we conclude the proof by showing that $\tilde{M}$ still admits a non vanishing ABC-Massey product and it still satisfies the $\partial\overline{\partial}$-lemma.

4.1 Cohomologies of complex orbifolds

In this section, we briefly recall the main facts about complex orbifolds and their cohomologies as proved classically in [22, 79, 128] and more recently in [9, 16, 135, 136], which will be needed later in this chapter. In particular, we will focus on complex orbifolds of global-quotient-type, which we will make use of in section 4.4.

The notion of orbifolds has been firstly introduced by Satake [128] under the name of *V-manifolds* and later studied by many authors, among the others, by Baily [22]. For the sake of completeness, we start by recalling the definition of a complex orbifold, following [22, Section 2].

Definition 4.1.1. Let $\hat{M}$ be a Hausdorff space and let $\tilde{U}$ be an open subset of $\hat{M}$. A *local uniformizing system*, shortly *l.u.s.*, for $\tilde{U}$ is a triple $\{U, G, \psi\}$ such that

- $U \subset \mathbb{C}^n$ is a connected open neighborhood of the origin of $\mathbb{C}^n$,
- G is a finite group of biholomorphisms of U,
- $\psi : U \to \tilde{U}$ is a continuous map such that $\psi \circ \sigma = \psi$, for every $\sigma \in G$, and the induced map of U/G onto $\tilde{U}$ is a homeomorphism.

Let us now consider two l.u.s.'s $\{U, G, \psi\}$ and $\{U', G', \psi'\}$ for, respectively $\tilde{U}$ and $\tilde{U}'$, open subsets of $\hat{M}$ such that $\tilde{U} \subset \tilde{U}'$. A biholomorphisms $\lambda : U \to U'$ is an *injection of* $\{U, G, \psi\}$ *into* $\{U', G', \psi'\}$ if, for any $\sigma \in G$, there exists $\sigma' \in G'$ satisfying the relations

$$\lambda \circ \sigma = \sigma' \circ \lambda,$$

$$\psi = \psi' \circ \lambda.$$

We recall the definition of complex orbifold.

Definition 4.1.2. A *complex orbifold* is a connected Hausdorff space $\hat{M}$ and a family $\mathcal{F}$ of l.u.s.'s for open subsets of $\hat{M}$ such that

- If $\{U, G, \psi\}, \{U', G', \psi'\} \in \mathcal{F}$ and $\tilde{U} = \psi(U) \subset \tilde{U}' = \psi(U')$, then there exists an injection of $\{U, G, \psi\}$ into $\{U', G', \psi'\}$;
- The open sets $\tilde{U}$ for which there exists a l.u.s. $\{U, G, \psi\} \in \mathcal{F}$ form a basis of open sets in $\hat{M}$.

Let $\mathcal{F}$ be the family of l.u.s.'s for open subsets of a complex orbifold $\hat{M}$. Then a *complex differential form* θ *on* $\hat{M}$ is defined to be a collection of complex differential forms $\{\theta_U\}$ on U which are G-invariant for $\{U, G, \psi\} \in \mathcal{F}$ and such that if $\lambda : \{U', G', \psi'\} \to \{U, G, \psi\}$ is an injection, we have that

$$\lambda^* \theta_U = \theta_{U'}.$$

Tensors such as vector fields and metrics on a complex orbifold $\hat{M}$ are similarly defined.

Let us then consider the graded complex of complex forms on the complex orbifold $\hat{M}$, namely, $(\wedge^\bullet_{\mathbb{C}} \hat{M}, d)$, and its associated bigraded complex $(\wedge^{\bullet,\bullet} \hat{M}, \overline{\partial}, \partial)$. As recalled in Section 1.3 for the usual cohomologies of manifolds, we can define *de Rham*, *Dolbeault*, *Bott-Chern*, and *Aeppli orbifold cohomologies* as

$$H^{p,q}_{dR}(\hat{M}) = \tfrac{\operatorname{Ker} d}{\operatorname{Im} d} \cap \wedge^{p,q}(\hat{M}), \qquad H^{p,q}_{\overline{\partial}}(\hat{M}) = \frac{\operatorname{Ker} \overline{\partial}}{\operatorname{Im} \overline{\partial}} \cap \wedge^{p,q}(\hat{M}), \tag{4.1.1}$$

$$H^{p,q}_{BC}(\hat{M}) = \tfrac{\operatorname{Ker} \partial \cap \operatorname{Ker} \overline{\partial}}{\operatorname{Im} \partial\overline{\partial}} \cap \wedge^{p,q}(\hat{M}), \qquad H^{p,q}_{A}(\hat{M}) = \frac{\operatorname{Ker} \partial\overline{\partial}}{\operatorname{Im} \partial + \operatorname{Im} \overline{\partial}} \cap \wedge^{p,q}(\hat{M}). \tag{4.1.2}$$

Starting from the complexes $(\bigwedge^\bullet \hat{M}, d)$ and $(\bigwedge^{\bullet,\bullet} \hat{M}, \partial, \overline{\partial})$, a spectral sequence $\{(E_r^\bullet, d_r)\}$ can be defined, so that $E_1^\bullet \simeq H_{\overline{\partial}}^{\bullet,\bullet}(\hat{M})$. From such sequence, known as *Hodge and Frölicher spectral sequence* of $\hat{M}$, one can derive the Frölicher inequality

$$\sum_{p+q=k} \dim_{\mathbb{C}} H_{\overline{\partial}}^{p,q}(\hat{M}) \geq \dim_{\mathbb{C}} H_{dR}^k(\hat{M};\mathbb{C}). \tag{4.1.3}$$

A complex orbifold is said to *satisfy the* $\partial\overline{\partial}$*-lemma* if the natural map $H_{BC}^{p,q}(\hat{M}) \to H_{\overline{\partial}}^{p,q}(\hat{M})$ is injective. Among many other characterizations, such property is equivalent, for a complex orbifold, to equality holding in equation (4.1.3) and to have isomorphisms induced by conjugation in Dolbeault cohomology, i.e.,

$$\overline{H_{\overline{\partial}}^{p,q}(\hat{M})} \simeq H_{\overline{\partial}}^{q,p}(\hat{M}), \tag{4.1.4}$$

see [46].

Once we fix an Hermitian metric g on a compact complex orbifold $\hat{M}$ of complex dimension n, one can define the $\mathbb{C}$-antilinear Hodge $*$-operator

$$*\colon \textstyle\bigwedge^{p,q} \hat{M} \to \bigwedge^{n-p,n-q} \hat{M},$$

the operators

$$d^* = - * d*, \quad \partial^* = - * \partial *, \quad \overline{\partial}^* = - * \overline{\partial} *,$$

the de Rham Laplacians Δ, Dolbeault Laplacian $\Delta_{\overline{\partial}}$, Bott-Chern Laplacian $\hat{\Delta}_{BC}$, and Aeppli Laplacian $\hat{\Delta}_A$ and their kernels

$$\mathcal{H}^k(\hat{M}, g) = \{\alpha \in \textstyle\bigwedge^k \hat{M} : \Delta\alpha = 0\},$$
$$\mathcal{H}_\sharp^{p,q}(\hat{M}, g) = \{\alpha \in \textstyle\bigwedge^{p,q} \hat{M} : \Delta_\sharp \alpha = 0\}, \quad \text{for} \quad \sharp \in \{\overline{\partial}, BC, A\}.$$

Harmonic forms on $\hat{M}$ with respect to each Laplacian can be characterized as in section 1.3 in equations (1.3.6), (1.3.7), and (1.3.8).

For a compact complex orbifold, the following theorem holds, see [128, Theorem 1],[22, Theorem K].

Theorem 4.1.3. *Let $\hat{M}$ be a compact complex orbifold of complex dimension n and g a Hermitian metric on $\hat{M}$. The following isomorphisms hold*

$$H_{dR}^k(\hat{M};\mathbb{C}) \to \mathcal{H}_{dR}^k(\hat{M}, g)$$
$$H_{\overline{\partial}}^{p,q}(\hat{M}) \to \mathcal{H}_{\overline{\partial}}^{p,q}(\hat{M}, g).$$

Moreover, the Hodge $*$-operator yields, respectively, the isomorphisms

$$H_{dR}^k(\hat{M}, \mathbb{C}) \simeq H_{dR}^{2n-k}(\hat{M}, \mathbb{C})$$
$$H_{\overline{\partial}}^{p,q}(\hat{M}) \simeq H_{\overline{\partial}}^{n-p,n-q}(\hat{M}).$$

Let us now consider the following class of complex orbifolds.

Let M be a complex manifold and G a finite subgroup of the group of biholomorphisms of M. If we consider the quotient

$$\hat{M} = M/G,$$

it turns out that, by the Bochner linearization theorem, see [28, Theorem 1], the space $\hat{M}$ is an orbifold as in the definition by Baily.

Definition 4.1.4. Orbifolds costructed in this way are said to be of *global-quotient-type*.

For compact orbifolds of global-quotient-type, besides Theorem 4.1.3, also Bott-Chern and Aeppli cohomologies can be computed in terms of harmonic representatives, as in the following theorem.

Theorem 4.1.5. *Let $\hat{M}$ be a compact complex orbifold of global-quotient type and g a Hermitian metric on $\hat{M}$. Then, the following isomorphisms hold*

$$H^{p,q}_{BC}(\hat{M}) \to \mathcal{H}^{p,q}_{BC}(\hat{M}, g)$$
$$H^{p,q}_{A}(\hat{M}) \to \mathcal{H}^{p,q}_{A}(\hat{M}, g).$$

In particular, the Hodge $*$-operator induces the isomorphisms

$$H^{p,q}_{BC}(\hat{M}) \simeq H^{n-p,n-q}_{A}(\hat{M}). \tag{4.1.5}$$

We conclude this section by recalling the property of the pull-back map of a proper surjective morphism of compact complex orbifolds, see [9].

Theorem 4.1.6. *Let $\hat{M}$ and $\hat{N}$ be compact complex orbifolds of the same complex dimension, and let $\pi\colon \hat{M} \to \hat{N}$ be a proper surjective morphism of complex orbifolds. Then the map $\pi\colon \hat{M} \to \hat{N}$ induces injective morphisms*

$$\pi^*_{dR}\colon H^k_{dR}(\hat{N}) \to H^k_{dR}(\hat{M})$$
$$\pi^*_{\overline{\partial}}\colon H^{p,q}_{\overline{\partial}}(\hat{N}) \to H^{p,q}_{\overline{\partial}}(\hat{M})$$
$$\pi^*_{BC}\colon H^{p,q}_{BC}(\hat{N}) \to H^{p,q}_{BC}(\hat{M}).$$

4.2 Dolbeault formalities are not closed

In this section we state and prove the non closedness result for the Dolbeault formalities as defined in section 1.4.

Throughout this section, we will denote by D the unit disc centered in the origin of $\mathbb{C}$, i.e., $D = \{z \in \mathbb{C} : |z| < 1\}$.

We recall that, by definition of a property closed under holomorphic deformations, for our purposes it will suffice to show the existence of a holomorphic family of compact complex manifolds $\{M_t\}_{t\in D}$ such that each M_t is geometrically Dolbeault formal and Dolbeault formal for $t \in D \smallsetminus \{0\}$, but M_0 has a non vanishing Dolbeault-Massey triple product. Given the relations (1.4.2) and (1.4.3) of section 1.4, this will assure that each M_t is also weakly-Dolbeault formal and every triple Dolbeault-Massey product is vanishing, for $t \in D \smallsetminus \{0\}$, but M_0 is neither weakly-Dolbeault formal, Dolbeault formal, nor geometrically Dolbeault formal, yielding the following result.

Theorem 4.2.1 ([132])**.** *The property of being geometrically Dolbeault formal, Dolbeault formal, weakly Dolbeault formal, and the vanishing of Dolbeault-Massey triple products are not closed under holomorphic deformations.*

In order to prove Theorem 4.2.1, we will provide a family $\{Z_t\}_{t\in D}$ of holomorphic deformations of the holomorphically parallelizable Nakamura manifold such that Z_t is geometrically Dolbeault formal and Dolbeault formal, for $t \neq 0$, but Z_0 has a non-trivial Dolbeault-Massey triple product.

To this purpose, let us start by considering the 6-dimensional simply-connected solvable Lie group G, with Lie algebra $\mathfrak{g}$ defined by the following structure equations of the frame

$\{e^1, e^2, e^3, e^4, e^5, e^6\}$ of $\mathfrak{g}^*$

$$\begin{cases} de^1 = e^{16} - e^{25}, \\ de^2 = e^{15} + e^{26}, \\ de^3 = -e^{36} + e^{45}, \\ de^4 = -e^{35} - e^{46}, \\ de^5 = de^6 = 0. \end{cases} \tag{4.2.1}$$

We then consider the holomorphically parallelizable complex structure on $\mathfrak{g}^*$. Define the almost complex structure on $\mathfrak{g}^*$, which we will denote by $J_{(0,0)}$, by setting

$$J_{(0,0)}e^1 = -e^2, \quad J_{(0,0)}e^2 = e^1, \quad J_{(0,0)}e^3 = -e^4, \quad J_{(0,0)}e^4 = e^3, \quad J_{(0,0)}e^5 = e^6, \quad J_{(0,0)}e^6 = -e^5.$$

Therefore, the following complex forms

$$\eta^1_{(0,0)} = e^1 + i\, e^2, \qquad \eta^2_{(0,0)} = e^3 + i\, e^4, \qquad \eta^3_{(0,0)} = \frac{1}{2}(e^5 - i\, e^6),$$

form a basis of (1,0)-forms for $(\mathfrak{g}^*)^{1,0}$ whose complex structure equations are

$$d\eta^1_{(0,0)} = 2i\, \eta^{13}_{(0,0)}, \qquad d\eta^2_{(0,0)} = -2i\, \eta^{23}_{(0,0)}, \qquad d\eta^3_{(0,0)} = 0. \tag{4.2.2}$$

According to Nakamura [105, p.90], G admits discrete uniform subgroups Γ, hence $M = \Gamma\backslash G$ is a complex 3-dimensional holomorphically parallelizable solvmanifold. If we fix any discrete uniform subgroup Γ of G, it turns out that $\{\eta^1_{(0,0)}, \eta^2_{(0,0)}, \eta^3_{(0,0)}\}$ is a global frame of left-invariant $(1,0)$-complex forms on M.

We note that the relations between the choice of discrete uniform subgroups of the Nakamura holomorphically parallelizable manifold and the dimensions of the Dolbeault and Bott-Chern cohomologies have been studied, for example, in [105, 14]. In particular, for every discrete uniform subgroup Γ, the left-invariant $(0,1)$-form $\eta^{\bar{3}}_{(0,0)}$ defines a non-zero Dolbeault cohomology class on the compact complex manifold $M_{(0,0)} = (\Gamma\backslash G, J_{(0,0)})$, where we denote $\eta^{\bar{j}}_{(0,0)} := \overline{\eta^{j}_{(0,0)}}$. Hence, we can use the class $[\eta^{\bar{3}}_{(0,0)}] \in H^{0,1}_{\bar{\partial}}(M_{(0,0)})$ to construct an appropriate holomorphic family of deformations.

Let $\mathbf{B} = \mathbb{C} \times D \subset \mathbb{C}^2$. For any $\mathbf{t} = (t_1, t_2) \in \mathbf{B}$, we set

$$\eta^1_{\mathbf{t}} := \eta^1_{(0,0)} + t_1\, \eta^{\bar{3}}_{(0,0)}, \quad \eta^2_{\mathbf{t}} := \eta^2_{(0,0)}, \quad \eta^3_{\mathbf{t}} := \eta^3_{(0,0)} + t_2\, \eta^{\bar{3}}_{(0,0)}. \tag{4.2.3}$$

Denote by $J_{\mathbf{t}}$ the left-invariant almost complex structure on $\mathfrak{g}$ associated to the coframe $\{\eta^1_{\mathbf{t}}, \eta^2_{\mathbf{t}}, \eta^3_{\mathbf{t}}\}$. It follows that $J_{\mathbf{t}}$ gives rise to an almost complex structure on $\Gamma\backslash G$. A direct computation shows that the structure equations of the $(1,0)$-forms $\{\eta^1_{\mathbf{t}}, \eta^2_{\mathbf{t}}, \eta^3_{\mathbf{t}}\}$ are

$$\begin{cases} d\eta^1_{\mathbf{t}} &= \frac{2i}{1-|t_2|^2}\, \eta^{13}_{\mathbf{t}} - \frac{2it_2}{1-|t_2|^2}\, \eta^{1\bar{3}}_{\mathbf{t}} + \frac{2it_1}{1-|t_2|^2}\, \eta^{3\bar{3}}_{\mathbf{t}}, \\ d\eta^2_{\mathbf{t}} &= -\frac{2i}{1-|t_2|^2}\, \eta^{23}_{\mathbf{t}} + \frac{2it_2}{1-|t_2|^2}\, \eta^{2\bar{3}}_{\mathbf{t}}, \\ d\eta^3_{\mathbf{t}} &= 0. \end{cases} \tag{4.2.4}$$

Let us set $M_{\mathbf{t}} := (\Gamma\backslash G, J_{\mathbf{t}})$. Then, for any fixed $\mathbf{t} \in \mathbf{B}$, equations (4.2.4) imply that, for any given $\alpha_{\mathbf{t}} \in \mathcal{A}^{1,0}M_{\mathbf{t}}$,

$$d\alpha_{\mathbf{t}} \in \mathcal{A}^{2,0}M_{\mathbf{t}} \oplus \mathcal{A}^{1,1}M_{\mathbf{t}}.$$

Hence, $J_{\mathbf{t}}$ is integrable for any $\mathbf{t} \in \mathbf{B}$.

Therefore, for any $\mathbf{t} = (t_1, t_2) \in \mathbf{B}$, we have a left-invariant complex structure $J_{\mathbf{t}}$ on $\Gamma\backslash G$, and so a compact complex manifold $M_{\mathbf{t}}$ of complex dimension 3.

Before proceeding, we need the following result.

Lemma 4.2.2 ([132]). *If $t_1 \neq 0$ and $t_2 = 0$, then the compact complex manifold $M_{\mathbf{t}=(t_1,0)}$ has a non-vanishing Dolbeault-Massey triple product.*

Proof (of Lemma 4.2.2). Let us consider the Dolbeault cohomology classes $[\eta^3_{(t_1,0)}] \in H^{1,0}_{\bar\partial}(M_{(t_1,0)})$ and $[\eta^{\bar 3}_{(t_1,0)}] \in H^{0,1}_{\bar\partial}(M_{(t_1,0)})$. From (4.2.4) for $t_2 = 0$ and $t_1 \neq 0$, we have the following relations:

$$\eta^3_{(t_1,0)} \wedge \eta^3_{(t_1,0)} = 0, \qquad \eta^3_{(t_1,0)} \wedge \eta^{\bar 3}_{(t_1,0)} = \bar\partial\left(\frac{-i}{2t_1}\eta^1_{(t_1,0)}\right).$$

Hence, $\langle [\eta^3_{(t_1,0)}], [\eta^3_{(t_1,0)}], [\eta^{\bar 3}_{(t_1,0)}]\rangle$ is a Dolbeault-Massey triple product which is represented (up to a constant) by the (2,0)-form $\eta^1_{(t_1,0)} \wedge \eta^3_{(t_1,0)}$. This (2,0)-form obviously defines a non-zero Dolbeault cohomology class in $H^{2,0}_{\bar\partial}(M_{(t_1,0)})$. Now, for showing that the product is non-trivial, it remains to prove that the class $[\eta^1_{(t_1,0)} \wedge \eta^3_{(t_1,0)}]$ does not belong to the ideal $[\eta^3_{(t_1,0)}] \cdot H^{1,0}_{\bar\partial}(M_{(t_1,0)})$.

Suppose that $[\eta^1_{(t_1,0)} \wedge \eta^3_{(t_1,0)}] \in [\eta^3_{(t_1,0)}] \cdot H^{1,0}_{\bar\partial}(M_{(t_1,0)})$. Then, there exists a (1,0)-form α on the manifold $M_{(t_1,0)}$ satisfying $\bar\partial\alpha = 0$ and $\eta^1_{(t_1,0)} \wedge \eta^3_{(t_1,0)} = \alpha \wedge \eta^3_{(t_1,0)}$. Now, since the complex structure is left-invariant, we can apply the symmetrization process (it preserves the bidegree of the forms) to get an invariant (1,0)-form $\tilde\alpha$ which is $\bar\partial$-closed and satisfies $(\eta^1_{(t_1,0)} - \tilde\alpha) \wedge \eta^3_{(t_1,0)} = 0$. But from (4.2.4) for $t_2 = 0$ and $t_1 \neq 0$, it follows that $\tilde\alpha = \lambda\,\eta^2_{(t_1,0)} + \mu\,\eta^3_{(t_1,0)}$ for some constants $\lambda, \mu \in \mathbb{C}$ in order to be $\bar\partial$-closed, so the condition $(\eta^1_{(t_1,0)} - \tilde\alpha) \wedge \eta^3_{(t_1,0)} = 0$ cannot be satisfied. □

Proof of Theorem 4.2.1. Let us now fix any $t_1^0 \in \mathbb{C} \smallsetminus \{0\}$. For any $t_2 \in D$, we consider the left-invariant complex structure $J_{\mathbf{t}=(t_1^0,t_2)}$ on G. By (4.2.4) the complex structure equations are

$$\begin{cases} d\eta^1_{\mathbf{t}} &= \frac{2i}{1-|t_2|^2}\,\eta^{13}_{\mathbf{t}} - \frac{2it_2}{1-|t_2|^2}\,\eta^{1\bar 3}_{\mathbf{t}} + \frac{2it_1^0}{1-|t_2|^2}\,\eta^{3\bar 3}_{\mathbf{t}}, \\ d\eta^2_{\mathbf{t}} &= -\frac{2i}{1-|t_2|^2}\,\eta^{23}_{\mathbf{t}} + \frac{2it_2}{1-|t_2|^2}\,\eta^{2\bar 3}_{\mathbf{t}}, \\ d\eta^3_{\mathbf{t}} &= 0. \end{cases} \tag{4.2.5}$$

If we take any $t_2 \in D \smallsetminus \{0\}$, we consider the basis $\{\tau^1_{\mathbf{t}}, \tau^2_{\mathbf{t}}, \tau^3_{\mathbf{t}}\}$ of $(1,0)$-forms with respect to $J_{\mathbf{t}}$ defined by

$$\tau^1_{\mathbf{t}} := 2i\,\eta^3_{\mathbf{t}}, \quad \tau^2_{\mathbf{t}} := \eta^1_{\mathbf{t}} + \frac{t_1^0}{t_2}\,\eta^3_{\mathbf{t}}, \quad \tau^3_{\mathbf{t}} := \eta^2_{\mathbf{t}}.$$

It is easy to check with respect to this basis, the complex structure equations become

$$\begin{cases} d\tau^1_{\mathbf{t}} &= 0, \\ d\tau^2_{\mathbf{t}} &= -\frac{1}{1-|t_2|^2}\,\tau^{12}_{\mathbf{t}} + \frac{t_2}{1-|t_2|^2}\,\tau^{2\bar 1}_{\mathbf{t}}, \\ d\tau^3_{\mathbf{t}} &= \frac{1}{1-|t_2|^2}\,\tau^{13}_{\mathbf{t}} - \frac{t_2}{1-|t_2|^2}\,\tau^{3\bar 1}_{\mathbf{t}}. \end{cases} \tag{4.2.6}$$

In [13] it is proved that there is a family of lattices $\{\Gamma_{t_2}\}_{t_2 \in D}$ on the Lie group G such that the compact manifold $\Gamma_{t_2}\backslash G$ endowed with the complex structure $\{J_{(t_1^0,t_2)}\}_{t_2\in D}$ given by (4.2.6) satisfies the $\partial\bar\partial$-lemma for any $t_2 \in D \smallsetminus \{0\}$, and, therefore, is Dolbeault formal. Indeed, notice that the equations (4.2.6) are precisely the complex equations found in [13, Table 3] for the holomorphic deformation (C_1) in [13, Proposition 4.2].

Also, it is easy to check that the harmonic representatives of Dolbeault cohomology listed in [13, Table 3] with respect to the canonical metric have a structure of algebra with respect to $\wedge$, therefore $M_{\mathbf{t}}$ is also geometrically Dolbeault formal.

Hence, we consider the following holomorphic family of compact complex manifolds $\{Z_t\}_{t\in D}$. Let us fix any $t_1^0 \in \mathbb{C} \setminus \{0\}$ and consider $t = t_2$ for $t \in D$. We take the previous lattices $\Gamma_t := \Gamma_{t_2}$ on the Lie group G given in [13] and the (left-invariant) complex structure $J_t = J_{(t_1^0, t)}$ on G, to obtain the family of compact complex manifolds $\{Z_t\} = \{(\Gamma_t \backslash G, J_t)\}$.

As we pointed out above, each compact complex manifold Z_t is Dolbeault formal and geometrically Dolbeault formal for any $t \neq 0$. However, the central fiber Z_0 has a non-vanishing Dolbeault-Massey triple product by Lemma 4.2.2, since this result holds for any lattice of maximal rank in G, in particular for the given lattice Γ. □

4.3 Bott-Chern formality is not closed

In this section, we prove the non closedness result for geometrically-Bott-Chern-formal manifolds and the vanishing of Aeppli-Bott-Chern-Massey products.

As for Dolbeault formality in section 4.2, it suffices to show the existence of a holomorphic family of compact complex manifolds $\{M_t\}_{t\in D}$, $D = \{z \in \mathbb{C} : |z| < 1\}$, such that M_t is geometrically-Bott-Chern formal for $t \in D \setminus \{0\}$, but M_0 admits a non-vanishing Aeppli-Bott-Chern-Massey triple product. In fact, by Proposition 1.4.10, M_t is geometrically-Bott-Chern formal and also has no non-vanishing Aeppli-Bott-Chern-Massey triple products, whereas M_0 would be not geometrically-Bott-Chern formal, thus proving the following result.

Theorem 4.3.1 ([132]). *The property of being geometrically-Bott-Chern-formal and the vanishing of Aeppli-Bott-Chern-Massey triple products are not closed under holomorphic deformations.*

In order to prove Theorem 4.3.1, we will use a different representation of the Nakamura holomorphically parallelizable manifolds by choosing a suitable family of lattices.

Let $(M = \Gamma\backslash G, J)$ be the Nakamura holomorphically parallelizable manifold, where

- $G := \mathbb{C} \ltimes \mathbb{C}^2$ is the solvable complex Lie group defined by $\gamma(z_1) * (z_2, z_3) = (e^{-z_1} z_2, e^{z_1} z_3)$;
- $\Gamma := (a\mathbb{Z} + 2\pi i\mathbb{Z}) \ltimes \Gamma''$ is a lattice of G of maximal rank, with Γ'' a lattice of $\mathbb{C}^2$;
- J is the holomorphically parallelizable complex structure on M induced by the natural standard complex structure on $\mathbb{C}^3 \simeq \mathbb{C} \ltimes \mathbb{C}^2$.

In particular, we point out that with this choice of Γ, it holds that $h^{0,1}_{\overline{\partial}}(M, J) = 3$ (see [14]) and a basis of invariant $(1,0)$-forms is given by $\{\eta^1 := dz^1, \eta^2 := e^{-z_1} dz^2, \eta^3 := e^{z_1} dz^3\}$ whose structure equations are

$$d\eta^1 = 0, \quad d\eta^2 = -\eta^{12}, \quad d\eta^3 = \eta^{13}. \tag{4.3.1}$$

Since $[\eta^{\bar{1}}] \in H^{0,1}_{\overline{\partial}}(M)$ is a non-zero cohomology class, we can consider the deformation constructed in [13] given by the $(0,1)$-vector form $\varphi(t)$ as follows

$$\varphi(t) := t \frac{\partial}{\partial z^1} \otimes \eta^{\bar{1}}, \qquad t \in D.$$

The resulting almost complex structure J_t is then characterized by the following coframe of $(1,0)$-forms on (M, J_t)

$$\begin{cases} \eta_t^1 := \eta^1 + t\eta^{\bar{1}} \\ \eta_t^2 := \eta^2 \\ \eta_t^3 := \eta^3, \end{cases}$$

whose structure equations are

$$\begin{cases} d\eta_t^1 &= 0, \\ d\eta_t^2 &= -\frac{1}{1-|t|^2}\,\eta_t^{12} + \frac{t}{1-|t|^2}\,\eta_t^{2\bar{1}}, \\ d\eta_t^3 &= \frac{1}{1-|t|^2}\,\eta_t^{13} - \frac{t}{1-|t|^2}\,\eta_t^{3\bar{1}}. \end{cases} \tag{4.3.2}$$

It is clear that J_t is integrable, thus giving rise to the holomorphic family of compact complex manifolds (M, J_t), for every $t \in D$.

Let us fix on M_t the Hermitian metric g_t whose fundamental form is $\omega_t = \frac{i}{2}(\eta_t^{1\bar{1}}+\eta_t^{2\bar{2}}+\eta_t^{3\bar{3}})$. Then, as proved in [13], for every $t \neq 0$, the manifold (M, J_t) satisfies the $\partial\overline{\partial}$-lemma and the harmonic representatives of the Bott-Chern cohomology of (M, J_t) for $t \neq 0$ are as in Table 4.4.16.

It is easy to check that $\mathcal{H}^{\bullet,\bullet}_{BC}(M, g_t)$ has a structure of algebra induced by the $\wedge$ product of forms. Therefore, the manifolds (M, J_t) are all geometrically Bott-Chern formal for $t \neq 0$.

Proof (of Theorem 4.3.1). It will suffices to construct a non zero Aeppli-Bott-Chern Massey triple product on $(M, J_0) = (M, J)$.

As proved in [14], the harmonic representatives of the Bott-Chern cohomology of (M, J) with respect to the canonical diagonal metric g are as listed in Table 4.4.17.

As a first remark, we notice that $\mathcal{H}^{\bullet,\bullet}_{BC}(M, g)$ does not have a structure of algebra induced by the $\wedge$ product of form. In fact, the product $\eta^{12} \wedge \left(e^{\bar{z}_1 - z_1}\eta^{3\bar{1}}\right)$ is not harmonic with respect to the Bott-Chern Laplacian, since

$$e^{\bar{z}_1 - z_1}\eta^{123\bar{1}} = \partial\overline{\partial}(-e^{\bar{z}_1 - z_1}\eta^{23}).$$

Therefore, take the following Bott-Chern cohomology classes

$$\mathfrak{a} := [\eta^{12}]_{BC}, \quad \mathfrak{b} = [e^{\bar{z}_1 - z_1}\eta^{3\bar{1}}]_{BC}, \quad \mathfrak{c} := [\eta^{\overline{12}}]_{BC}. \tag{4.3.3}$$

Since $\mathfrak{a} \cup \mathfrak{b} = [e^{\bar{z}_1 - z_1}\eta^{123\bar{1}}] = 0 \in H^{3,1}_{BC}(M)$ and clearly $\mathfrak{b} \cup \mathfrak{c} = 0 \in H^{1,3}_{BC}(M)$, by Definition 1.4.8 we obtain that

$$[e^{\bar{z}_1 - z_1}\eta^{23\overline{12}}]_A \in \frac{H^{2,2}_A(M)}{[\eta^{12}]_{BC} \cup H^{0,2}_A(M) + [\eta^{\overline{12}}]_{BC} \cup H^{2,0}_A(M)} \tag{4.3.4}$$

is the Aeppli-Bott-Chern-Massey triple product $\langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \rangle_{ABC}$.

We proceed by showing that, as a cohomology class, $[e^{\bar{z}_1 - z_1}\eta^{23\overline{12}}]_A \neq 0$. Indeed, it can be easily seen from structure equations (4.3.1) that the form $e^{\bar{z}_1 - z_1}\eta^{23\overline{12}}$ is $\partial\overline{\partial}$-closed and, since $*\left(e^{\bar{z}_1 - z_1}\eta^{23\overline{12}}\right) = e^{z_1 - \bar{z}_1}\eta^{\overline{13}}$, it is a light matter of computations to show that

$$\partial * \left(e^{\bar{z}_1 - z_1}\eta^{23\overline{12}}\right) = 0, \quad \overline{\partial} * \left(e^{\bar{z}_1 - z_1}\eta^{23\overline{12}}\right) = 0.$$

Therefore, conditions (1.3.11) assure that $e^{\bar{z}_1 - z_1}\eta^{23\overline{12}}$ is Δ_A-harmonic and therefore, as a Aeppli cohomology class, $[e^{\bar{z}_1 - z_1}\eta^{23\overline{12}}]_A \neq 0$. Actually, from Table 4.4.17, one can directly compute the spaces $H^{2,2}_A(M)$, $H^{2,0}_A(M)$, and $H^{0,2}_A(M)$, by the relations $H^{p,q}_A(M) = *\left(H^{n-p,n-q}_{BC}(M)\right)$, obtaining

$$\begin{aligned} H^{2,0}_A(M) =& \mathbb{C}\langle [e^{z_1 - \bar{z}_1}\eta^{12}], [e^{\bar{z}_1 - z_1}\eta^{13}], [\eta^{23}] \rangle, \\ H^{2,2}_A(M) =& \mathbb{C}\langle [e^{z_1 - \bar{z}_1}\eta^{12\overline{13}}], [e^{z_1 - \bar{z}_1}\eta^{12\overline{23}}], [e^{\bar{z}_1 - z_1}\eta^{13\overline{12}}], \\ & [e^{\bar{z}_1 - z_1}\eta^{13\overline{23}}], [e^{\bar{z}_1 - z_1}\eta^{23\overline{12}}], [e^{z_1 - \bar{z}_1}\eta^{23\overline{13}}], [\eta^{23\overline{23}}] \rangle, \\ H^{0,2}_A(M) =& \mathbb{C}\langle [e^{\bar{z}_1 - z_1}\eta^{\overline{12}}], [e^{z_1 - \bar{z}_1}\eta^{\overline{13}}], [e^{\overline{23}}] \rangle, \end{aligned}$$

in which we displayed the Δ_A-harmonic representatives with respect to the canonical diagonal metric g on (M, J).

It remains to show that $[e^{\overline{z}_1 - z_1}\eta^{23\overline{12}}]_A \notin [\eta^{12}]_{BC} \cup H_A^{0,2}(M) + [\eta^{\overline{12}}]_{BC} \cup H_A^{2,0}(M)$.

We point out that a generic element $\mathfrak{d} \in [\eta^{12}]_{BC} \cup H_A^{0,2}(M) + [\eta^{\overline{12}}]_{BC} \cup H_A^{2,0}(M)$ can be written as

$$\mathfrak{d} = [Ae^{\overline{z}_1 - z_1}\eta^{12\overline{12}} + Be^{z_1 - \overline{z}_1}\eta^{12\overline{13}} + C\eta^{12\overline{23}} A' e^{z_1 - \overline{z}_1}\eta^{13\overline{13}} + B' e^{\overline{z}_1 - z_1}\eta^{13\overline{12}} + C'\eta^{23\overline{12}}]_A,$$

for $A, B, C, A', B', C' \in \mathbb{C}$.

By contradiction, let us suppose that

$$[e^{\overline{z}_1 - z_1}\eta^{23\overline{12}}]_A = [Ae^{\overline{z}_1 - z_1}\eta^{12\overline{12}} + Be^{z_1 - \overline{z}_1}\eta^{12\overline{13}} + C\eta^{12\overline{23}} A' e^{z_1 - \overline{z}_1}\eta^{13\overline{13}} + B' e^{\overline{z}_1 - z_1}\eta^{13\overline{12}} + C'\eta^{23\overline{12}}]_A,$$

for some $A, B, C, A', B', C' \in \mathbb{C}$, or equivalently, by definition of Aeppli cohomology, that

$$e^{\overline{z}_1 - z_1}\eta^{23\overline{12}} = Ae^{\overline{z}_1 - z_1}\eta^{12\overline{12}} + Be^{z_1 - \overline{z}_1}\eta^{12\overline{13}} + C\eta^{12\overline{23}} A' e^{z_1 - \overline{z}_1}\eta^{13\overline{13}} + B' e^{\overline{z}_1 - z_1}\eta^{13\overline{12}} + C'\eta^{23\overline{12}} + \partial\lambda + \overline{\partial}\mu, \tag{4.3.5}$$

for some forms $\lambda \in \mathcal{A}^{1,2}(M)$, $\mu \in \mathcal{A}^{2,1}(M)$.

However, we observe that the following forms are ∂ or $\overline{\partial}$ exact, i.e.,

$$\begin{cases} \eta^{12\overline{23}} = \partial(\eta^{2\overline{23}}) \\ e^{\overline{z}_1 - z_1}\eta^{12\overline{12}} = \partial(-\frac{1}{2}e^{\overline{z}_1 - z_1}\eta^{2\overline{12}}) \\ \eta^{23\overline{12}} = \overline{\partial}(-\eta^{23\overline{2}}) \\ e^{z_1 - \overline{z}_1}\eta^{13\overline{13}} = \overline{\partial}(-\frac{1}{2}e^{z_1 - \overline{z}_1}\eta^{13\overline{3}}), \end{cases}$$

therefore equation (4.3.5) reduces to

$$e^{\overline{z}_1 - z_1}\eta^{23\overline{12}} = Be^{z_1 - \overline{z}_1}\eta^{12\overline{13}} + B' e^{\overline{z}_1 - z_1}\eta^{13\overline{12}} + \partial\lambda + \overline{\partial}\mu. \tag{4.3.6}$$

In particular, since $e^{\overline{z}_1 - z_1}\eta^{23\overline{12}}, e^{z_1 - \overline{z}_1}\eta^{12\overline{13}}, e^{\overline{z}_1 - z_1}\eta^{13\overline{12}} \in \mathcal{H}_A^{2,2}(M, g)$, it must hold that

$$e^{\overline{z}_1 - z_1}\eta^{23\overline{12}} - (Be^{z_1 - \overline{z}_1}\eta^{12\overline{13}} + B' e^{\overline{z}_1 - z_1}\eta^{13\overline{12}}) \in \mathcal{H}_A^{2,2}(M, g)$$

therefore, equation (4.3.6) boils down to

$$e^{\overline{z}_1 - z_1}\eta^{23\overline{12}} - Be^{z_1 - \overline{z}_1}\eta^{12\overline{13}} - B' e^{\overline{z}_1 - z_1}\eta^{13\overline{12}} = 0,$$

for some $B, B' \in \mathbb{C}$, but this clearly cannot hold. Thus, we obtain a contradiction and hence

$$[e^{\overline{z}_1 - z_1}\eta^{23\overline{12}}]_A \notin [\eta^{12}]_{BC} \cup H_A^{0,2}(M) + [\eta^{\overline{12}}]_{BC} \cup H_A^{2,0}(M),$$

showing that $\langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \rangle$ defines a non vanishing Aeppli-Bott-Chern-Massey triple product on (M, J).

By Proposition 1.4.10, we can conclude that (M, J) is also not geometrically-Bott-Chern formal.

□

4.4 Aeppli-Bott-Chern-Massey products and the $\partial\overline{\partial}$-lemma

In this section, we show that the Aeppli-Bott-Chern-Massey triple products are not an obstruction for the $\partial\overline{\partial}$-lemma on a compact complex manifold, unlike Massey triple products and Dolbeault-Massey triple products, see [106, Theorem 8]. In fact, we will costruct a global-quotient-type complex orbifold by taking the quotient of the Iwasawa manifold with respect to the action of a finite group of biholomorphisms and we will prove that it satisfies the $\partial\overline{\partial}$-lemma but it admits a non-vanishing Aeppli-Bott-Chern-Massey triple product. As a final step, we will we will construct a smooth non-Kähler resolution of such complex orbifold still satisfying the $\partial\overline{\partial}$-lemma and admitting a non-vanishing Aeppli-Bott-Chern-Massey triple product.

Theorem 4.4.1 ([132])**.** *There exists a compact complex manifold satisfying the $\partial\overline{\partial}$-lemma and admitting a non-vanishing ABC-Massey triple product.*

We start by considering the complex 3-dimensional Heisenberg group $G := \mathbb{H}(3,\mathbb{C})$, i.e., the nilpotent group of matrices

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

As an open set of $GL(n;\mathbb{C})$, G has standard holomorphic coordinates $\{z_1, z_2, z_3\}$.

If we take the lattice $\Gamma = G \cap GL(3;\mathbb{Z}[i])$, the compact quotient $M = \Gamma\backslash G$ is a complex nilmanifold of complex dimension 3, the *Iwasawa manifold.*

The group G admits a left-invariant coframe of $(1,0)$-forms

$$\varphi^1 = dz_1, \quad \varphi^2 = dz_2, \quad \varphi^3 = dz_3 - z_1 dz_2$$

which gives rise to a left-invariant integrable almost complex structure J on G.

We note that the coframe $\{\varphi^1, \varphi^2, \varphi^3\}$, and therefore the complex structure J, descends on the quotient M. Since the structure equations on (M, J) are

$$d\varphi^1 = 0, \quad d\varphi^2 = 0, \quad d\varphi^3 = -\varphi^{12}, \tag{4.4.1}$$

the complex structure J is holomorphically parallelizable on M. Therefore, by [11, Theorem 2.8], we know that de Rham cohomology, Dolbeault cohomology, Bott-Chern cohomology and Aeppli cohomology of (M, J) are isomorphic to the corresponding cohomologies of the Lie algebra $\mathfrak{g}$ of G endowed with the complex structure J.

We point out that the Iwasawa manifold does not satisfy the $\partial\overline{\partial}$-lemma. In fact, it is not formal [69].

We now construct an orbifold of global-quotient-type starting from M. We first define the following action $\sigma\colon \mathbb{C}^3 \to \mathbb{C}^3$ by

$$\sigma(z_1, z_2, z_3) = (iz_1, iz_2, -z_3), \quad \text{for} \quad (z_1, z_2, z_3) \in \mathbb{C}^3. \tag{4.4.2}$$

We observe that as a group of biholomorphisms $\langle\sigma\rangle$ has finite order, since $\sigma^4 = \mathrm{id}_{\mathbb{C}^3}$.

We need the following.

Lemma 4.4.2. *The action σ is well defined on M.*

Proof. We begin by noting that G can be identified with $(\mathbb{C}^3, \star)$, where the product $\star$ is given by

$$(z_1, z_2, z_3) \star (w_1, w_2, w_3) = (z_1 + w_1, z_2 + w_2, z_3 + z_1 w_2 + w_3) \tag{4.4.3}$$

for every $(z_1, z_2, z_3), (w_1, w_2, w_3) \in \mathbb{C}^3$.

We then need to show that, for $[z], [z'] \in M$, if $[z] = [z']$, then $[\sigma(z)] = [\sigma(z')]$, or, equivalently, that if $z = (z_1, z_2, z_3) \sim z' = (z'_1, z'_2, z'_3)$, then $\sigma(z) \sim \sigma(z')$.

The equivalence is given by the action of multiplication on the left by elements of Γ, which, through the identification $G \simeq (\mathbb{C}^3, \star)$ reads $z \sim z'$ if, and only if, there exists $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{Z}[i])^3$ such that $z' = \gamma \star z$, which accounts to

$$\begin{cases} z'_1 = z_1 + \gamma_1 \\ z'_2 = z_2 + \gamma_2 \\ z'_3 = z_3 + \gamma_1 z_2 + \gamma_3. \end{cases} \tag{4.4.4}$$

Let us then assume that $z \sim z'$. We point out that

$$\sigma(z') = (iz'_1, iz'_2, -z'_3)$$

and, by (4.4.4),

$$\sigma(z') = (iz_1 + i\gamma_1, iz_2 + i\gamma_2, -z_3 - \gamma_1 z_2 - \gamma_3). \tag{4.4.5}$$

Now choose $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3) := (i\gamma_1, i\gamma_2, -\gamma_3) \in (\mathbb{Z}[i])^3$. By definition (4.4.3) of the product $\star$ and equation (4.4.5), it is easy to check that

$$\sigma(z') = \tilde{\gamma} \star \sigma(z).$$

□

As a consequence of Lemma 4.4.2, we can define an action of σ on M, given by $\sigma([z]) := [\sigma(z)]$, for every $[z] \in M$.

Let us now consider the quotient $M/\langle\sigma\rangle$. It is not a smooth manifold, as follows from the following lemma.

Lemma 4.4.3. *The action σ on M has 16 fixed points.*

Proof. We need to find all the solution to the following equation

$$\sigma[z] = [z], \quad \text{for} \quad z = (z_1, z_2, z_3) \in \mathbb{C}^3, \tag{4.4.6}$$

or, equivalently, to $\sigma(z) \sim z$, i.e., finding all the distinct solutions (up to equivalence) to

$$\begin{cases} iz_1 = z_1 + \gamma_1 \\ iz_2 = z_2 + \gamma_2 \\ -z_3 = z_3 + \gamma_1 z_2 + z_3, \end{cases} \tag{4.4.7}$$

for $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (\mathbb{Z}[i])^3$. Now, by writing $z_j = x_j + iy_j$ and $\gamma_j = m_j + ik_j$, the system (4.4.7) yields the following solutions

$$\begin{cases} z_1 = \frac{1}{2}(-m_1 + k_1 + i(-m_1 - k_1)) \\ z_2 = \frac{1}{2}(-m_2 + k_2 + i(-m_2 - k_2)) \\ z_3 = \frac{1}{4}(m_1 m_2 - k_1 k_2 - m_1 k_2 - k_1 m_2 - 2m_3 + i(m_1 m_2 - k_1 k_2 + m_1 k_2 + k_1 m_2 - 2k_3)). \end{cases} \tag{4.4.8}$$

We observe that two points in $z = (z_1, z_2, z_3), z' = (z'_1, z'_2, z'_3) \in \mathbb{C}^3$ satisfying (4.4.8) are equivalent in $(\mathbb{Z}[i])\backslash\mathbb{C}^3$ if, and only if, there exists $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{Z}[i])^3$ such that $z' = \lambda \star z$, i.e.,

$$\begin{cases} z'_1 = z_1 + \lambda_1 \\ z'_2 = z_2 + \lambda_2 \\ z'_3 = z_3 + \lambda_1 z_2 + \lambda_3. \end{cases} \tag{4.4.9}$$

We look at the first equation. By writing each $\lambda_j = a_j + ib_j$ and using (4.4.8), we have that $z'_1 - z_1 = \lambda_1$ if, and only if,

$$\frac{1}{2}(-m'_1 + k'_1 - (-m_1 + k_1)) = a_1$$
$$\frac{1}{2}(-m'_1 - k'_1 - (m_1 - k_1)) = b_1.$$

We notice that $[-m_1 - k_1] = [-m_1 + k_1] \in \frac{\mathbb{Z}}{2\mathbb{Z}}$. Therefore $z_1' - z_1 = \lambda_1$ if and only if $[-m_1 + k_1] = [-m_1' + k_1'] \in \frac{\mathbb{Z}}{2\mathbb{Z}}$. By choosing as representatives for γ_1 as either 0 or 1, we obtain that the distinct values of z_1, up to equivalence, are either 0 or $\frac{1}{2} + \frac{i}{2}$. Analogously, this can be done for z_2, whose distinct values, up to equivalence, are 0 or $\frac{1}{2} + \frac{i}{2}$. By plugging those values in the third equation of (4.4.8), we get that, in the case where $(z_1, z_2), (z_1', z_2') \neq (\frac{1}{2} + \frac{i}{2}, \frac{1}{2} + \frac{i}{2})$, the third components of z and z' are, respectively, $z_3 = -\frac{1}{2}m_3 - \frac{i}{2}k_3$ and $z_3' = -\frac{1}{2}m_3' - \frac{i}{2}k_3'$. Then, equation $z_3' - z_3 = \lambda_1 z_2 + \lambda_3$ is satisfied if and only if

$$\begin{aligned}\frac{1}{2}(m_3 - m_3') &= a_3\\ \frac{1}{2}(k_3 - k_3') &= b_3.\end{aligned}$$

Hence, by choosing $\gamma_3 \in \{0, 1, i, 1 + i\}$, we get that the only solutions, up to equivalence, are $z_3 \in \{0, \frac{1}{2}, \frac{i}{2}, \frac{1}{2} + \frac{i}{2}\}$.

Finally, when $z_1 = z_1' = z_2 = z_2' = \frac{1}{2} + \frac{i}{2}$, we have expression for $z_3 = \frac{1}{4}(1 - 2m_3 + i(1 - 2k_3))$ and $z_3' = \frac{1}{4}(1 - 2m_3' + i(1 - 2k_3'))$ Therefore, $z_3' - z_3 = \lambda_1 z_2 + \lambda_3$ holds if, and only if,

$$\begin{aligned}\frac{1}{2}(m_3 - m_3') &= a_1 + a_3\\ \frac{1}{2}(k_3 - k_3') &= b_1 + b_3.\end{aligned}$$

Thus, if one chooses $\gamma_3 \in \{0, 1, i, 1 + i\}$, one gets that the solutions, up to equivalence, are $z_3 \in \{0, \frac{1}{2}, \frac{i}{2}, \frac{1}{2} + \frac{i}{2}\}$. By counting all the distinct solutions up to equivalence $z = (z_1, z_2, z_3)$ satisfying (4.4.6), i.e., the fixed point of σ on M, we find that they are 16 and, clearly, isolated. □

As consequence of Lemma 4.4.3, we obtain that $\hat{M} := M/\langle\sigma\rangle$ is a singular orbifold of global-quotient-type. Since

$$\sigma^*\varphi^1 = i\varphi^1, \quad \sigma^*\varphi^2 = i\varphi^2, \quad \sigma^*\varphi^3 = -\varphi^3,$$

the complex of σ-invariant differential forms on M is

$$\bigwedge\nolimits^{\bullet,\bullet}\hat{M} = \mathrm{Span}_{\mathbb{C}}\left\langle 1, \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}}, \varphi^{123}, \varphi^{12\bar{3}}, \varphi^{3\bar{1}\bar{2}}, \varphi^{\bar{1}\bar{2}\bar{3}}, \varphi^{12\bar{1}\bar{2}}, \varphi^{13\bar{1}\bar{3}}, \varphi^{13\bar{2}\bar{3}}, \varphi^{23\bar{1}\bar{3}}, \varphi^{23\bar{2}\bar{3}}, \varphi^{123\bar{1}\bar{2}\bar{3}}\right\rangle.$$

Let us fix g the Hermitian metric on $\hat{M}$ with fundamental associated form $\omega = \frac{i}{2}(\varphi^{1\bar{1}} + \varphi^{2\bar{2}} + \varphi^{3\bar{3}})$. We can now compute the cohomologies of $\hat{M}$ by definitions (4.1.1) and (4.1.2) and via Theorems 4.1.3 and 4.1.5. In particular, we prove the following.

Lemma 4.4.4. *$\hat{M}$ satisfies the $\partial\overline{\partial}$-lemma.*

Proof. It suffices to the show that Frölicher equality (4.1.3) holds and also $H^{p,q}_{\overline{\partial}}(\hat{M}) \simeq H^{q,p}_{\overline{\partial}}(\hat{M})$ via complex conjugation. By easy computations of the harmonic representatives with respect to g, we see that the non-trivial de Rham cohomology spaces of $\hat{M}$ are

$$\begin{aligned}H^0_{dR}(\hat{M}; \mathbb{C}) &= \mathrm{Span}_{\mathbb{C}}\langle 1\rangle\\ H^2_{dR}(\hat{M}; \mathbb{C}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}}\rangle\\ H^3_{dR}(\hat{M}; \mathbb{C}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{123}, \varphi^{\bar{1}\bar{2}\bar{3}}\rangle\\ H^4_{dR}(\hat{M}; \mathbb{C}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{13\bar{1}\bar{3}}, \varphi^{13\bar{2}\bar{3}}, \varphi^{23\bar{1}\bar{3}}, \varphi^{23\bar{2}\bar{3}}\rangle\\ H^6_{dR}(\hat{M}; \mathbb{C}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{123\bar{1}\bar{2}\bar{3}}\rangle,\end{aligned}$$

whereas the non-trivial Dolbeault cohomology spaces of $\hat{M}$ are

$$\begin{aligned}
H^{0,0}_{\overline{\partial}}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle 1\rangle\\
H^{1,1}_{\overline{\partial}}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{1\overline{1}}, \varphi^{1\overline{2}}, \varphi^{2\overline{1}}, \varphi^{2\overline{2}}\rangle\\
H^{3,0}_{\overline{\partial}}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{123}\rangle\\
H^{0,3}_{\overline{\partial}}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{\overline{123}}\rangle\\
H^{2,2}_{\overline{\partial}}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{13\overline{13}}, \varphi^{13\overline{23}}, \varphi^{23\overline{13}}, \varphi^{23\overline{23}}\rangle\\
H^{3,3}_{\overline{\partial}}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{123\overline{123}}\rangle.
\end{aligned}$$

By comparing the former and the latter spaces, we easily conclude the proof. □

As a consequence, Bott-Chern and Aeppli cohomologies of $\hat{M}$ are immediately determined by $H^{p,q}_{BC}(\hat{M}) = H^{p,q}_{\overline{\partial}}(\hat{M})$ and $H^{p,q}_{A}(\hat{M}) \simeq *(H^{3-p,3-q}_{BC}(\hat{M}))$, yielding

$$\begin{aligned}
H^{0,0}_{BC}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle 1\rangle & H^{0,0}_{A}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle 1\rangle\\
H^{1,1}_{BC}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{1\overline{1}}, \varphi^{1\overline{2}}, \varphi^{2\overline{1}}, \varphi^{2\overline{2}}\rangle & H^{1,1}_{A}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{1\overline{1}}, \varphi^{1\overline{2}}, \varphi^{2\overline{1}}, \varphi^{2\overline{2}}\rangle\\
H^{3,0}_{BC}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{123}\rangle & H^{3,0}_{A}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{123}\rangle\\
H^{0,3}_{BC}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{\overline{123}}\rangle & H^{0,3}_{A}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{\overline{123}}\rangle\\
H^{2,2}_{BC}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{13\overline{13}}, \varphi^{13\overline{23}}, \varphi^{23\overline{13}}, \varphi^{23\overline{23}}\rangle & H^{2,2}_{A}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{13\overline{13}}, \varphi^{13\overline{23}}, \varphi^{23\overline{13}}, \varphi^{23\overline{23}}\rangle\\
H^{3,3}_{BC}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{123\overline{123}}\rangle & H^{3,3}_{A}(\hat{M}) &= \mathrm{Span}_{\mathbb{C}}\langle \varphi^{123\overline{123}}\rangle.
\end{aligned}$$

We now define an ABC-Massey triple product on $\hat{M}$.

Lemma 4.4.5. *$\hat{M}$ admits a non vanishing ABC-Massey triple product.*

Proof. Let us consider the following Bott-Chern cohomology classes

$$[\alpha] := [\varphi^{1\overline{1}}] \in H^{1,1}_{BC}(\hat{M}), \quad [\beta] := [\varphi^{2\overline{2}}] \in H^{1,1}_{BC}(\hat{M}), \quad [\gamma] := [\varphi^{2\overline{2}}] \in H^{1,1}_{BC}(\hat{M}).$$

We notice that, by structure equations (4.4.1), we have that $\varphi^{1\overline{1}} \wedge \varphi^{2\overline{2}} = \partial\overline{\partial}\varphi^{3\overline{3}}$. Then, it is well-defined

$$\langle [\alpha], [\beta], [\gamma]\rangle_{ABC} \in \frac{H^{2,2}_{A}(\hat{M})}{[\varphi^{1\overline{1}}]_{BC} \cup H^{1,1}_{A}(\hat{M}) + [\varphi^{2\overline{2}}]_{BC} \cup H^{1,1}_{A}(\hat{M})},$$

which, by Definition 1.4.8, is represented by the non zero Aeppli cohomology class $[\varphi^{23\overline{23}}] \in H^{2,2}_{A}(\hat{M})$. By the previous description of Aeppli cohomology, we note that the ideal $[\varphi^{1\overline{1}}]_{BC} \cup H^{1,1}_{A}(\hat{M}) + [\varphi^{2\overline{2}}]_{BC} \cup H^{1,1}_{A}(\hat{M})$ is actually trivial in $H^{2,2}_{A}(\hat{M})$.

Hence, $\langle [\varphi^{1\overline{1}}], [\varphi^{2\overline{2}}], [\varphi^{2\overline{2}}]\rangle_{ABC}$ is a non-vanishing ABC-Massey triple product on $\hat{M}$. □

Proof (of Theorem 4.4.1). (I) In view of Hironaka singularities resolutions theorem, see [72], it turns out that $\hat{M}$ admits a resolution.
We will construct an explicit smooth resolution $\hat{M}$, proceedings as follows, see [36]. Define $\psi = \sigma^2$, i.e.,

$$\psi(z_1, z_2, z_3) = (-z_1, -z_2, z_3)$$

for every $(z_1, z_2, z_3) \in \mathbb{C}^3$. Clearly, ψ descends to M and has order 2 on M, i.e., since $\psi^2 = \mathrm{id}_M$. The locus of fixed points by the action of ψ on M is the disjoint union of 8 curves on M given by

$$\mathcal{C}_i = \{[z_1^0, z_2^0, z_3] : z_3 \in \mathbb{C}\},$$

with $(z_1^0, z_2^0) \in \{(0, \frac{1}{2}), (0, \frac{i}{2}), (0, \frac{1}{2} + \frac{i}{2}), (\frac{1}{2}, 0), (\frac{i}{2}, 0), (\frac{1}{2} + \frac{i}{2}, 0), (\frac{1}{2} + \frac{i}{2}, \frac{1}{2} + \frac{i}{2})\}$.

Let us set $\mathcal{C} := \mathcal{C}_1 = \{[0, 0, z_3]\}$. In a neighborhood U of $\mathcal{C}$ and local coordinates (z_1, z_2, z_3), we write, locally, $\mathcal{C} = \{(z_1 = 0, z_2 = 0, z_3)\}$. We perform the blow-up of M along $\mathcal{C}$ by taking the set

$$\tilde{U} = \{((z_1, z_2, z_3), [l_1 : l_2]) : z_1 l_2 - z_2 l_1 = 0\} \subset U \times \mathbb{P}^2.$$

Through the resulting the map $p \colon Bl_{\mathcal{C}} M \to M$, if $E := p^{-1}(\mathcal{C}) \simeq \mathbb{P}(\mathcal{N}_{\mathcal{C}/M})$ is the exceptional divisor, $\tilde{U} \smallsetminus E$ projects biholomorphically onto $U \smallsetminus \mathcal{C}$.

On $\tilde{U}_1 = \{l_1 \neq 0\}$, we have that $z_2 = \frac{l_2}{l_2} z_1$ and local coordinates on $\tilde{U}_1$ are given by

$$\zeta_1^{(1)} = z_1, \quad \zeta_2^{(1)} = \frac{l_2}{l_1}, \quad \zeta_3^{(1)} = z_3,$$

whereas on $\tilde{U}_2 = \{l_2 \neq 0\}$, we have that $z_1 = \frac{l_1}{l_2} z_2$ and the following

$$\zeta_1^{(2)} = \frac{l_1}{l_2}, \quad \zeta_2^{(2)} = z_2, \quad \zeta_3^{(2)} = z_3$$

are local coordinates on $\tilde{U}_2$. In the following, we will show the procedure only on $\tilde{U}_1$ since on $\tilde{U}_2$ the approach is analogous.

Notice that ψ induces a morphism $\tilde{\psi}$ on $Bl_{\mathcal{C}} M$. In particular, we have that, on $\tilde{U}_1$,

$$\tilde{\psi}(\zeta_1^{(1)}, \zeta_2^{(1)}, \zeta_3^{(1)}) = (-\zeta_1^{(1)}, \zeta_2^{(1)}, \zeta_3^{(1)}).$$

Let us then consider the quotient $M' = Bl_{\mathcal{C}} M / \langle \tilde{\psi} \rangle$. On the quotient $\tilde{U}_1 / \langle \tilde{\psi} \rangle \subset M'$, the action σ' induced by σ acts as

$$\sigma'([\zeta_1^{(1)}, \zeta_2^{(1)}, \zeta_3^{(1)}]_{\tilde{\psi}}) = [i\zeta_1^{(1)}, \zeta_2^{(1)}, -\zeta_3^{(1)}]_{\tilde{\psi}}. \tag{4.4.10}$$

Note that, through local coordinates, $\tilde{U}_1 / \langle \tilde{\psi} \rangle$ is identified with with $\mathbb{C}^3 / \langle \tilde{\psi} \rangle$. So we construct local coordinates for the latter in the following way. The holomorphic map $f \colon \mathbb{C}^3 \to \mathbb{C}^3$ defined by

$$f(w_1, w_2, w_3) = (w_1^2, w_2, w_3), \quad \text{for } (w_1, w_2, w_3) \in \mathbb{C}^3,$$

which on local coordinates on $\tilde{U}_1$ acts as

$$f(\zeta_1^{(1)}, \zeta_2^{(1)}, \zeta_3^{(1)}) = ((\zeta_1^{(1)})^2, \zeta_2^{(1)}, \zeta_3^{(1)}),$$

gives rise to the following diagram

$$\begin{tikzcd}
\mathbb{C}^3_{(\zeta_1^{(1)}, \zeta_2^{(1)}, \zeta_3^{(1)})} \arrow[r, "f"] \arrow[d] & \mathbb{C}^3_{(w_1, w_2, w_3)} \\
\mathbb{C}^3 / \langle \tilde{\psi} \rangle \arrow[ur, "\hat{f}"'] &
\end{tikzcd}$$

where $\hat{f}([\zeta^{(1)}]_{\tilde{\psi}}) := f(\zeta^{(1)})$ is well defined and, in fact, a biholomorphism.

Therefore, we can identify $\tilde{U}_1/\langle\tilde{\psi}\rangle$ with $\mathbb{C}^3_{(w_1,w_2,w_3)}$. We now look for fixed point of σ' on M'. Locally, we must then consider the action of σ', which on $\mathbb{C}^3_{(w_1,w_2,w_3)}$ acts as $\tilde{\sigma} := \hat{f}^{-1}\circ\sigma'\circ\hat{f}$. Recalling equation (4.4.10), we see that

$$\tilde{\sigma}(w_1,w_2,w_3) = (-w_1,w_2,-w_3) \tag{4.4.11}$$

for any $(w_1,w_2,w_3)\in\mathbb{C}^3$, yielding that the locus of the fixed points of σ' on $\tilde{U}_1/\langle\tilde{\psi}\rangle$ is given, locally, by the set

$$\mathcal{D} = \{(w_1,w_2,w_3)\} = \{(0,w_2,0)\}.$$

We now perform the further blow-up $p' \colon Bl_{\mathcal{D}}M' \to M'$, by considering

$$\tilde{\tilde{U}}^{(1)} = \{((w_1,w_2,w_3),[v_1:v_3]) : w_1v_3 - w_3v_1 = 0\}.$$

On $\tilde{\tilde{U}}^{(1)}_1 := \{v_1 \neq 0\}$, we have that $w_3 = \frac{v_3}{v_1}w_1$ and local coordinates are given by

$$\eta^{(1)}_1 = w_1, \quad \eta^{(1)}_2 = w_2, \quad \eta^{(1)}_3 = \frac{v_3}{v_1}, \tag{4.4.12}$$

whereas on $\tilde{\tilde{U}}^{(1)}_3 := \{v_3 \neq 0\}$, we have that $w_1 = \frac{v_1}{v_3}w_3$ and the following

$$\eta^{(3)}_1 = \frac{v_1}{v_3}, \quad \eta^{(3)}_2 = w_2, \quad \eta^{(3)}_3 = w_3, \tag{4.4.13}$$

are local coordinates on $\tilde{\tilde{U}}^{(1)}_3$.

We now study the quotient $Bl_{\mathcal{D}}M'$ by the induced action of $\langle\tilde{\sigma}'\rangle$. By recalling the local action of $\tilde{\sigma}$ (4.4.11) and the expressions (4.4.12) and (4.4.13) for local coordinates, on $\tilde{\tilde{U}}^{(1)}_1$, we have that

$$\tilde{\sigma}'(\eta^{(1)}_1,\eta^{(1)}_2,\eta^{(1)}_3) = (-\eta^{(1)}_1,\eta^{(1)}_2,\eta^{(1)}_3),$$

whereas on $\tilde{\tilde{U}}^{(1)}_3$, we have that

$$\tilde{\sigma}'(\eta^{(3)}_1,\eta^{(3)}_2,\eta^{(3)}_3) = (\eta^{(3)}_1,\eta^{(3)}_2,-\eta^{(3)}_3),$$

i.e.,

$$\tilde{\tilde{U}}^{(1)}_1/\langle\tilde{\sigma}'\rangle \simeq \frac{\mathbb{C}}{\pm\,\mathrm{id}} \times \mathbb{C}^2_{(\eta^{(1)}_2,\eta^{(1)}_3)}$$

and

$$\tilde{\tilde{U}}^{(1)}_3/\langle\tilde{\sigma}'\rangle \simeq \frac{\mathbb{C}}{\pm\,\mathrm{id}} \times \mathbb{C}^2_{(\eta^{(3)}_1,\eta^{(3)}_2)}.$$

Hence, since $\tilde{\tilde{U}}^{(1)}_1/\langle\tilde{\sigma}'\rangle$ and $\tilde{\tilde{U}}^{(1)}_3/\langle\tilde{\sigma}'\rangle$ are smooth manifolds, the manifold $Bl_{\mathcal{D}}M'/\langle\tilde{\sigma}'\rangle$ is smooth.

As mentioned before, the same procedure can be applied starting from $\tilde{U}_2$, which results in finding smooth resolutions of the singular points in the chart $\tilde{U}_2 \subset Bl_{\mathcal{C}}M$.
Therefore, if we denote by $\tilde{M}_1$ the resulting complex manifold and the projection $p'' \colon \tilde{M}_1 \to M/\langle\sigma\rangle$, we obtain a smooth resolution of the fixed curve $\mathcal{C} = \mathcal{C}_1$ on $M/\langle\sigma\rangle$.

By repeating the analogous procedure for every fixed locus $\mathcal{C}_i$, we obtain a smooth resolution

$$\pi \colon \hat{M} \to \tilde{M},$$

as the diagram summarizes

$$\begin{tikzcd}
M \arrow[d] & Bl_{Fix_\psi} M \arrow[l] \arrow[d] & \\
M/\langle\psi\rangle \arrow[d] & M' := Bl_{Fix_\psi} M/\langle\tilde{\psi}\rangle \arrow[l] \arrow[d] & Bl_{Fix_{\tilde{\sigma}}} M' \arrow[l] \arrow[d] \\
\hat{M} := M/\langle\sigma\rangle & M'/\langle\sigma'\rangle \arrow[l] & Bl_{Fix_{\tilde{\sigma}}} M'/\langle\tilde{\sigma}'\rangle. \arrow[l]
\end{tikzcd}$$

(II) We now show the following:

(i) $\tilde{M}$ admits a non-vanishing ABC-Massey triple product;

(ii) $\tilde{M}$ satisfies the $\partial\overline{\partial}$-lemma.

(i) We proceed by considering the pull-back through π of the Bott-Chern cohomology classes used in Lemma 4.4.5, i.e., we consider the classes $[\pi^*\varphi^{1\overline{1}}] \in H^{1,1}_{BC}(\tilde{M})$ and $[\pi^*\varphi^{2\overline{2}}] \in H^{1,1}_{BC}(\tilde{M})$. They are well-defined and non-vanishing, by Theorem 4.1.6. Since $\pi^*(\varphi^{1\overline{1}}) \wedge \pi^*(\varphi^{2\overline{2}}) = \partial\overline{\partial}(\pi^*\varphi^{3\overline{3}})$, the ABC-Massey product

$$\langle[\pi^*\varphi^{1\overline{1}})], [\pi^*\varphi^{2\overline{2}}], [\pi^*\varphi^{2\overline{2}}]\rangle_{ABC} \in \frac{H^{2,2}_A(\tilde{M})}{[\pi^*\varphi^{1\overline{1}}]_{BC} \cup H^{1,1}_A(\tilde{M}) + [\pi^*\varphi^{2\overline{2}}]_{BC} \cup H^{1,1}_A(\tilde{M})}$$

is well-defined and represented by $[\pi^*\varphi^{23\overline{23}}] \in H^{2,2}_A(\tilde{M})$. Again, by Theorem 4.1.6, this class is not vanishing.

It remains to show that

$$[\pi^*\varphi^{23\overline{23}}]_A \notin [\pi^*\varphi^{1\overline{1}}]_{BC} \cup H^{1,1}_A(\tilde{M}) + [\pi^*\varphi^{2\overline{2}}]_{BC} \cup H^{1,1}_A(\tilde{M}).$$

By contradiction, let us suppose the converse, i.e.,

$$[\pi^*\varphi^{23\overline{23}}]_A = [\pi^*\varphi^{1\overline{1}}]_{BC} \cup [F]_A + [\pi^*\varphi^{2\overline{2}}]_{BC} \cup [G]_A, \tag{4.4.14}$$

for some $[F], [G] \in H^{1,1}_A(\tilde{M})$. Let us now multiply by $[\pi^*\varphi^{1\overline{1}}]_{BC}$ each side of (4.4.14), to obtain

$$\begin{aligned}
[\pi^*\varphi^{123\overline{123}}]_A &= [\pi^*\varphi^{1\overline{1}} \wedge \pi^*\varphi^{2\overline{2}}]_{BC} \cup [G]_A \\
&= [\pi^*(\partial\overline{\partial}\varphi^{3\overline{3}})]_{BC} \cup [G]_A \\
&= [\partial\overline{\partial}(\pi^*\varphi^{3\overline{3}})]_{BC} \cup [G]_A \\
&= [\partial\overline{\partial}(\pi^*\varphi^{3\overline{3}} \wedge G)]_A = 0 \in H^{2,2}_A(\tilde{M}),
\end{aligned}$$

which leads to contradiction, since π^* is injective by Theorem 4.1.6 and $[\varphi^{123\overline{123}}]_A \neq 0$.

(ii) We now observe that the fixed points loci along which we perform the blow-ups are complex lines, which are naturally Kähler. Therefore, they satisfy the $\partial\overline{\partial}$-lemma. As proved in Lemma 4.4.4, also $\hat{M}$ satisfies the $\partial\overline{\partial}$-lemma. We can then apply [16, Theorem 25], to conclude that the resolution $\tilde{M}$ of $\hat{M}$ satisfies the $\partial\overline{\partial}$-lemma. □

Remark 4.4.6. Notice that the obtained manifold $\tilde{M}$ is not a Kähler manifold. Indeed, let us assume by contradiction the opposite, i.e., let us suppose there exists a Kähler metric $\tilde{g}$ on $\tilde{M}$ with fundamental form $\tilde{\omega}$. Then, by Stokes theorem and structure equations (4.4.1), we obtain

$$0 = \int_{\tilde{M}} d(\tilde{\omega} \wedge \pi^*(\varphi^{12\overline{3}})) = \int_{\tilde{M}} \tilde{\omega} \wedge d(\pi^*\varphi^{12\overline{3}}) = -\int_{\tilde{M}} \tilde{\omega} \wedge \pi^*\varphi^{12\overline{12}}. \tag{4.4.15}$$

However, since the form ω is transverse, the above integral vanishes if and only $\tilde{\omega} \wedge \pi^* \varphi^{12\overline{3}}$ is identically 0, if and only, $\pi^* \varphi^{12\overline{3}}$ is identically 0, which is a contradiction.

(p,q)	$H^{p,q}_{BC}(M, J_t), \quad t \in D \smallsetminus \{0\}$
$(0,0)$	$\mathbb{C}\langle 1 \rangle$
$(1,0)$	$\mathbb{C}\langle \eta_t^{1} \rangle$
$(0,1)$	$\mathbb{C}\langle \eta_t^{\overline{1}} \rangle$
$(2,0)$	$\mathbb{C}\langle \eta_t^{23} \rangle$
$(1,1)$	$\mathbb{C}\langle \eta_t^{1\overline{1}}, e^{z_1-\overline{z}_1}\eta_t^{2\overline{3}}, e^{\overline{z}_1-z_1}\eta_t^{3\overline{2}} \rangle$
$(0,2)$	$\mathbb{C}\langle \eta_t^{\overline{23}} \rangle$
$(3,0)$	$\mathbb{C}\langle \eta_t^{123} \rangle$
$(2,1)$	$\mathbb{C}\langle e^{z_1-\overline{z}_1}\eta_t^{12\overline{3}}, e^{\overline{z}_1-z_1}\eta_t^{13\overline{2}}, \eta_t^{23\overline{1}} \rangle$
$(1,2)$	$\mathbb{C}\langle e^{\overline{z}_1-z_1}\eta_t^{3\overline{12}}, e^{z_1-\overline{z}_1}\eta_t^{2\overline{13}}, \eta_t^{1\overline{23}} \rangle$
$(0,3)$	$\mathbb{C}\langle \eta_t^{\overline{123}} \rangle$
$(3,1)$	$\mathbb{C}\langle \eta_t^{123\overline{1}} \rangle$
$(2,2)$	$\mathbb{C}\langle e^{z_1-\overline{z}_1}\eta_t^{12\overline{12}}, e^{\overline{z}_1-z_1}\eta_t^{13\overline{12}}, \eta_t^{23\overline{23}} \rangle$
$(1,3)$	$\mathbb{C}\langle \eta_t^{1\overline{123}} \rangle$
$(3,2)$	$\mathbb{C}\langle \eta_t^{123\overline{23}} \rangle$
$(2,3)$	$\mathbb{C}\langle \eta_t^{23\overline{123}} \rangle$
$(3,3)$	$\mathbb{C}\langle \eta_t^{123\overline{123}} \rangle$

(4.4.16)

(p,q)	$H^{p,q}_{BC}(M,J)$
$(0,0)$	$\mathbb{C}\langle 1\rangle$
$(1,0)$	$\mathbb{C}\langle \eta^{1}\rangle$
$(0,1)$	$\mathbb{C}\langle \eta^{\overline{1}}\rangle$
$(2,0)$	$\mathbb{C}\langle \eta^{12},\eta^{13},\eta^{23}\rangle$
$(1,1)$	$\mathbb{C}\langle \eta^{1\overline{1}}, e^{\overline{z}_1-z_1}\eta^{1\overline{2}}, e^{z_1-\overline{z}_1}\eta^{1\overline{3}}, e^{z_1-\overline{z}_1}\eta^{2\overline{1}}, \eta^{z_1-\overline{z}_1}\eta^{2\overline{3}}, e^{\overline{z}_1-z_1}\eta^{3\overline{1}}, e^{\overline{z}_1-z_1}\eta^{3\overline{2}}\rangle$
$(0,2)$	$\mathbb{C}\langle \eta^{\overline{12}},\eta^{\overline{13}},\eta^{\overline{23}}\rangle$
$(3,0)$	$\mathbb{C}\langle \eta^{123}\rangle$
$(2,1)$	$\mathbb{C}\langle \eta^{12\overline{1}}, e^{z_1-\overline{z}_1}\eta^{12\overline{1}}, e^{\overline{z}_1-z_1}\eta^{12\overline{2}}, e^{z_1-\overline{z}_1}\eta^{12\overline{3}}, \eta^{13\overline{1}}, e^{\overline{z}_1-z_1}\eta^{13\overline{1}}, e^{\overline{z}_1-z_1}\eta^{13\overline{2}}, e^{z_1-\overline{z}_1}\eta^{13\overline{3}}, \eta^{23\overline{1}}\rangle$
$(1,2)$	$\mathbb{C}\langle \eta^{1\overline{12}}, e^{\overline{z}_1-z_1}\eta^{1\overline{12}}, \eta^{1\overline{13}}, e^{z_1-\overline{z}_1}\eta^{1\overline{13}}, \eta^{1\overline{23}}, e^{z_1-\overline{z}_1}\eta^{2\overline{12}}, e^{z_1-\overline{z}_1}\eta^{2\overline{13}}, e^{\overline{z}_1-z_1}\eta^{3\overline{12}}, e^{\overline{z}_1-z_1}\eta^{3\overline{13}}\rangle$
$(0,3)$	$\mathbb{C}\langle \eta^{\overline{123}}\rangle$
$(3,1)$	$\mathbb{C}\langle \eta^{123\overline{1}}, e^{\overline{z}_1-z_1}\eta^{123\overline{2}}, e^{z_1-\overline{z}_1}\eta^{123\overline{3}}\rangle$
$(2,2)$	$\mathbb{C}\langle e^{z_1-\overline{z}_1}\eta^{12\overline{12}}, e^{\overline{z}_1-z_1}\eta^{13\overline{12}}, \eta^{23\overline{23}}\rangle$
$(1,3)$	$\mathbb{C}\langle \eta^{1\overline{123}}, e^{z_1-\overline{z}_1}\eta^{2\overline{123}}, e^{\overline{z}_1-z_1}\eta^{3\overline{123}}\rangle$
$(3,2)$	$\mathbb{C}\langle \eta^{123\overline{12}}, e^{\overline{z}_1-z_1}\eta^{123\overline{12}}, \eta^{123\overline{13}}, e^{z_1-\overline{z}_1}\eta^{123\overline{13}}, \eta^{123\overline{23}}\rangle$
$(2,3)$	$\mathbb{C}\langle \eta^{12\overline{123}}, e^{z_1-\overline{z}_1}\eta^{12\overline{123}}, \eta^{13\overline{123}}, e^{\overline{z}_1-z_1}\eta^{13\overline{123}}, \eta^{23\overline{123}}\rangle$
$(3,3)$	$\mathbb{C}\langle \eta^{123\overline{123}}\rangle$

(4.4.17)

Chapter 5

Geometric formalities along the Chern-Ricci flow

In this chapter, we focus on geometric formalities of complex manifolds and their dependence on the Hermitian metric. In [150, 147], the authors study the behaviour of Dolbeault formality, respectively geometric-Bott-Chern formality, under small deformations of the complex structure. Here, we keep the complex structure fixed, and we study geometric formalities with respect to Hermitian metrics evolving along a geometric flow. More precisely, we consider the *Chern-Ricci flow* [63, 151] that evolves the fundamental form $\omega(t)$ of a Hermitian form by the Chern-Ricci form,

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}^{Ch}(\omega),$$

and we study the possible algebra structure on the space of (de Rham, Dolbeault, Bott-Chern, Aeppli) harmonic forms with respect to $\omega(t)$ varying t.

We study in details geometric formality according to Kotschick for a whole class of surfaces evolving by the Chern-Ricci flow, *i.e.* compact complex non-Kähler surfaces with Kodaira dimension $\mathrm{Kod}(X) = -\infty$ and first Betti number $b_1(X) = 1$, known as *class VII* of the Enriques-Kodaira classification. In particular, we first rule out class VII surfaces with second Betti number $b_2 > 0$ by applying arguments as in [87]. Then, we exploit the structure of quotients of Lie groups with invariant complex and Hermitian structure on the only class VII surfaces with $b_2 = 0$, that is Hopf and Inoue surfaces see [30, 83, 95, 148], in order to reduce the description of harmonic forms and the equation of the Chern-Ricci flow of such surfaces at the level of invariant forms and thus make explicit computations. We obtain Theorem 5.3.1. We also study the evolution of geometric formality according to Kotschick on other compact complex non-Kähler surfaces that are diffeomorphic to solvmanifolds, *e.g.* Kodaira surfaces. Since any complex structures on such surfaces is left-invariant, see [68, Theorem 1], we focus on invariant forms also in this case; we obtain Proposition 5.3.2. We note that, also in this case, it is possible to rule out primary Kodaira surfaces by the obstructions in [87] or [69], and therefore we focus on secondary Kodaira surfaces with initial invariant metrics.

Regarding Dolbeault and Bott-Chern geometric formalities evolving by the Chern-Ricci flow, by applying the analogous procedure on Hopf, Inoue, and Kodaira surfaces, we have reached results as follows. We also checked how the algebraic structures of Aeppli cohomology and its harmonic representatives are modified along the Chern-Ricci flow, obtaining Proposition 5.4.1.

Throughout this chapter, we give a complete description of harmonic forms on such compact complex surfaces depending on the invariant Hermitian metrics. We made computations with the aid of SageMath [117].

We ask whether for Dolbeault and Bott-Chern geometric formalities there exist obstructions (such as the ones found in [87]) which would help complete the picture for geometric formalities for class VII surfaces. We also ask whether the behaviour we observed is more general or there exist counterexamples.

5.1 Chern-Ricci flow on Hermitian manifolds

The Chern-Ricci flow (introduced in [63] and studied in [151]) is a parabolic geometric flow that preserves the Hermitian condition of the initial given metric. The equations that describe such flow on a Hermitian manifold (X^n, J, g_0) are

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}^{Ch}(\omega(t)), \qquad \omega(0) = \omega_0,$$

where ω_0, $\omega(t)$ are the foundamental forms associated, respectively, to the Hermitian initial metric g_0 and the evolution metric $g(t)$ by the usual relation $\omega(\cdot,\cdot) = g(J(\cdot),\cdot)$. For an arbitrary real $(1,1)$-form ω, $\mathrm{Ric}^{Ch}(\omega)$ is the Chern-Ricci form of ω. The first Chern-Ricci curvature Ric^{Ch} is defined starting from ∇^{Ch}, the Chern connection on (X, J, g), *i.e.* the unique connection ∇ on the holomorphic tangent bundle $T^{1,0}X$ such that ∇ is compatible with both g and J and $\nabla^{0,1} = \overline{\partial}$. In a holomorphic chart, the curvature tensor R^{Ch} of such connection has components $R^{Ch}_{i\bar{j}k\bar{l}}$, for $i, j, k, l \in \{1, \dots, n\}$. The Chern-Ricci tensor is obtained by contracting the last two indices via the metric

$$\mathrm{Ricci}^{Ch}_{i\bar{j}} := g^{k\bar{l}} R^{Ch}_{i\bar{j}k\bar{l}},$$

where $(g^{k\bar{l}})$ is the inverse of the matrix $(g_{k\bar{l}})$ representing in local coordinates the metric g. The Chern-Ricci form is defined by

$$\mathrm{Ric}^{Ch} := \mathrm{Ricci}^{Ch}(J(\,\cdot\,), \,\cdot\,).$$

Such form has important properties, among which a very simple form in local coordinates:

$$\mathrm{Ric}^{Ch}(\omega) = -\sqrt{-1}\partial\overline{\partial}\log\det(g),$$

from which we can deduce that $\mathrm{Ric}^{Ch}(\omega)$ is a ∂-, $\overline{\partial}$-closed form, hence it defines a cohomology class in $H^{1,1}_{BC}(X)$. Such class is a holomorphic invariant, denoted by $c_1^{BC}(X)$, which plays a fundamental role in the classification of complex manifolds.

5.2 Cohomology and Chern-Ricci flow on compact complex surfaces and quotients of Lie groups

In this section, we analyze in details complex structures, cohomologies, and Chern-Ricci flow on non-Kähler compact complex surfaces that can be described as quotients $M = H \backslash G$ of Lie groups G by a subgroup H, with M endowed with invariant complex structure J [68], namely Hopf, Inoue, and Kodaira surfaces.

Complex structure

As recalled in section 1.6, we can describe the complex structure J by a coframe of left-invariant $(1,0)$-forms $\{\varphi^1, \varphi^2\}$ on G and their conjugates, and by their structure equations

$$d\varphi^I = -c^I_{HK}\varphi^H \wedge \varphi^K,$$

equivalently, by the dual frame $\{\varphi_1, \varphi_2\}$ of $(1,0)$-vector fields and their conjugates, with structure equations $[\varphi_H, \varphi_K] = c^I_{HK}\varphi_I$. Note that here capital letters here vary in the ordered set $(1, 2, \bar{1}, \bar{2})$ and refer to the corresponding component. Moreover, the Einstein summation is assumed, for increasing indices in case of forms.

Hermitian structure

The arbitrary invariant Hermitian metric $g := \omega(\cdot, J(\cdot))$ on (M, J) has associated $(1,1)$-form

$$2\omega = \sqrt{-1}\sum_{I,J=1}^{2} g_{I\bar{J}}\,\varphi^I \wedge \varphi^{\bar{J}} = \sqrt{-1}r^2\,\varphi^{1\bar{1}} + \sqrt{-1}s^2\,\varphi^{2\bar{2}} + u\,\varphi^{1\bar{2}} - \bar{u}\,\varphi^{2\bar{1}} \tag{5.2.1}$$

where the coefficients satisfy

$$r^2 > 0, \qquad s^2 > 0, \qquad r^2 s^2 > |u|^2.$$

That is to say, the Hermitian matrix

$$(g_{KL})_{K,L} = \frac{1}{2}\cdot\begin{pmatrix} r^2 & -\sqrt{-1}u \\ \sqrt{-1}\bar{u} & s^2 \end{pmatrix} \in \mathrm{GL}(\mathfrak{g})$$

is positive-definite. Its inverse is

$$(g^{KL})_{K,L} := (g_{KL})^{-1}_{K,L} = \frac{2}{r^2s^2 - |u|^2}\cdot\begin{pmatrix} s^2 & \sqrt{-1}u \\ -\sqrt{-1}\bar{u} & r^2 \end{pmatrix}.$$

The Christoffel symbols of the Chern connection can be computed as follows, see *e.g.* [112]:

$$(\Gamma^{Ch})^K_{IH} = \frac{1}{2}c^K_{IH} - \frac{1}{2}g^{KA}g_{BI}c^B_{HA} - \frac{1}{2}g^{KA}g_{BH}c^B_{IA} + \frac{1}{2}g^{KL}C_{IHL},$$

where

$$C_{IHL} = d\omega(J\varphi_I, \varphi_H, \varphi_L).$$

We can then express the $(4,0)$-Riemannian curvature of the Chern connection as

$$(R^{Ch})_{IHKL} = g_{AL}(\Gamma^{Ch})^B_{HK}(\Gamma^{Ch})^A_{IB} - g_{AL}(\Gamma^{Ch})^B_{IK}(\Gamma^{Ch})^A_{HB} - g_{AL}c^B_{IH}(\Gamma^{Ch})^A_{BK},$$

and the Chern-Ricci tensor as

$$(\mathrm{Ricci}^{Ch})_{IH} = g^{KL}(R^{Ch})_{IHKL}.$$

Then the Chern-Ricci form is

$$\mathrm{Ric}^{Ch} = \mathrm{Ricci}^{Ch}(J(\cdot), \cdot) \in c_1^{BC}(X) \in H^{1,1}_{BC}(X;\mathbb{R}).$$

Finally, we collect here some explicit description of the Hodge-star-operator on forms for the arbitrary Hermitian metric associated to the form (5.2.1), in order to describe harmonicity, see also [116, Lemma 2]. It is straightforward to check that:

$$*_g\varphi^1 = \frac{\sqrt{-1}}{2}\bar{u}\varphi^{12\bar{1}} + \frac{1}{2}s^2\varphi^{12\bar{2}}, \qquad *_g\varphi^2 = -\frac{1}{2}r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar{2}}, \tag{5.2.2}$$

$$*_g\bar{\varphi}^1 = -\frac{\sqrt{-1}}{2}u\varphi^{1\bar{1}\bar{2}} + \frac{1}{2}s^2\varphi^{2\bar{1}\bar{2}}, \qquad *_g\bar{\varphi}^2 = -\frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}}; \tag{5.2.3}$$

$$*_g\varphi^{12} = \varphi^{12}, \qquad *_g\varphi^{\bar{1}\bar{2}} = \varphi^{\bar{1}\bar{2}} \tag{5.2.4}$$

$$V *_g \varphi^{1\bar{1}} = |u|^2\varphi^{1\bar{1}} - \sqrt{-1}us^2\varphi^{1\bar{2}} + \sqrt{-1}\bar{u}s^2\varphi^{2\bar{1}} + s^4\varphi^{2\bar{2}}, \tag{5.2.5}$$

$$V *_g \varphi^{1\bar{2}} = -\sqrt{-1}\bar{u}r^2\varphi^{1\bar{1}} - r^2s^2\varphi^{1\bar{2}} + \bar{u}^2\varphi^{2\bar{1}} - \sqrt{-1}\bar{u}s^2\varphi^{2\bar{2}}$$

$$V *_g \varphi^{2\bar{1}} = \sqrt{-1}ur^2\varphi^{1\bar{1}} + u^2\varphi^{1\bar{2}} - r^2s^2\varphi^{2\bar{1}} + \sqrt{-1}us^2\varphi^{2\bar{2}}$$

$$V *_g \varphi^{2\bar{2}} = r^4\varphi^{1\bar{1}} - \sqrt{-1}ur^2\varphi^{1\bar{2}} + \sqrt{-1}\bar{u}r^2\varphi^{2\bar{1}} + |u|^2\varphi^{2\bar{2}},$$

$$V *_g \varphi^{12\bar{1}} = -2\sqrt{-1}\,u\varphi^1 + 2\,s^2\varphi^2, \qquad V *_g \varphi^{12\bar{2}} = -2\,r^2\varphi^1 - 2\sqrt{-1}\,\bar{u}\varphi^2, \tag{5.2.6}$$

$$V *_g \varphi^{1\bar{1}\bar{2}} = 2\sqrt{-1}\,\bar{u}\bar{\varphi}^1 + 2\,s^2\bar{\varphi}^2, \qquad V *_g \varphi^{2\bar{1}\bar{2}} = -2\,r^2\bar{\varphi}^1 + 2\sqrt{-1}\,u\bar{\varphi}^2;$$

where we set $V = g^{1\bar{1}}g^{2\bar{2}} - g^{1\bar{2}}g^{2\bar{1}} = r^2s^2 - |u|^2$.

Cohomologies

Consider the inclusion of invariant forms into the double complex of forms,

$$\iota \colon (\wedge^{\bullet,\bullet}\mathfrak{g}^\vee, \partial, \overline{\partial}) \hookrightarrow (\wedge^{\bullet,\bullet}M, \partial, \overline{\partial}).$$

By choosing an invariant Hermitian metric, (the easier finite-dimensional version of) elliptic Hodge theory also applies at the level of invariant forms; in particular, any cohomology class of invariant forms admits a unique invariant harmonic representative. It follows that the above inclusion induces injective maps in de Rham ι_{dR}, Dolbeault $\iota_{\overline{\partial}}$, Bott-Chern ι_{BC}, Aeppli ι_A cohomology, see [41, Lemma 9]. We claim that they are in fact isomorphisms.

The de Rham cohomology of Hopf, Inoue, Kodaira surfaces is well known, and it happens that the above maps ι_{dR} are actually isomorphisms, that is, any de Rham class admits an invariant representative. In fact, the Hopf surface is diffeomorphic to the product $\mathbb{S}^1 \times \mathbb{S}^3$ of two compact Lie groups, so one can use the Künneth formula and *e.g.* [50, Theorem 1.28]; the primary Kodaira surface is a nilmanifold, so one can use the Nomizu theorem [108]; the secondary Kodaira surfaces are quotients of primary Kodaira surfaces by finite groups; the Inoue surface of type $S^{\pm}$ is a completely-solvable solvmanifold, so one can use the Hattori theorem [70]; and the de Rham cohomology of the Inoue surface of type S_M can be computed by exploiting their number-theoretic construction as [109] does in a more general setting.

As for the Dolbeault cohomology, for compact complex surfaces, we know that the Frölicher spectral sequence degenerates at the first page, see *e.g.* [23], that is,

$$\dim_{\mathbb{C}} H^k_{dR}(X;\mathbb{C}) = \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X)$$

for any k. By explicitly computing the Dolbeault cohomology of invariant forms [12], one then notice that the above maps $\iota_{\overline{\partial}}$ are actually isomorphisms.

Finally, Bott-Chern cohomology of compact complex surfaces is well-undestood since [149]. By [14, Theorem 1.3, Proposition 2.2], (that fits in the general theory later developed by [135],) also ι_{BC} are isomorphisms. Explicit computations can be found in [12]. Finally, ι_A are isomorphisms thanks to the Schweitzer duality between Bott-Chern and Aeppli cohomologies, where one can use the Hodge-star-operator with respect to an invariant Hermitian metric.

Finally, by uniqueness of the harmonic representative in a cohomology class, we also deduce that harmonic representatives with respect to invariant metrics are invariant.

Chern-Ricci flow

Recall that the Chern-Ricci form represents the first Bott-Chern class $c_1^{BC}(X) \in H^{1,1}_{BC}(X)$. Since a class in $H^{1,1}_{BC}(X)$ contains only one invariant representative, the Chern-Ricci form $\mathrm{Ric}^{Ch}(\omega)$ does not depend on the invariant Hermitian metric ω. In particular, the Chern-Ricci flow starting at ω_0 reduces to

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}^{Ch}(\omega_0), \qquad \omega(0) = \omega_0. \tag{CRF}$$

We notice that the solution of the Chern-Ricci flow starting at an invariant metric remains invariant for any existence time. Indeed, by short existence and uniqueness assured by parabolicity, the symmetry group is preserved along the flow (and possibly increases in the limit, see *e.g.* [93]).

Denote by ρ_r, ρ_s, ρ_u the coefficients of the Chern-Ricci form,

$$2\mathrm{Ric}^{Ch} = \sqrt{-1}\rho_r\varphi^{1\bar{1}} + \sqrt{-1}\rho_s\varphi^{2\bar{2}} + \rho_u\varphi^{1\bar{2}} - \bar{\rho}_u\varphi^{2\bar{1}},$$

and let the initial metric ω_0 be of the form

$$2\,\omega_0 = \sqrt{-1}r_0^2\,\varphi^{1\bar{1}} + \sqrt{-1}s_0^2\,\varphi^{2\bar{2}} + u_0\,\varphi^{1\bar{2}} - \bar{u}_0\,\varphi^{2\bar{1}}, \tag{5.2.7}$$

where $r_0, s_0 \in \mathbb{R} \smallsetminus \{0\}$ and $u_0 \in \mathbb{C}$ such that $r_0^2 s_0^2 - |u_0|^2 > 0$. The solution $\omega(t)$ of the Chern-Ricci flow starting at ω_0 is then

$$2\omega(t) = \sqrt{-1}(r_0^2 - t\rho_r)\varphi^{1\bar{1}} + \sqrt{-1}(s_0^2 - t\rho_s)\varphi^{2\bar{2}} + (u_0 - t\rho_u)\varphi^{1\bar{2}} - (\bar{u}_0 - t\bar{\rho}_u)\varphi^{2\bar{1}},$$

defined for times t such that $r(t)^2 = r_0^2 - t\rho_r > 0$, $s(t) = s_0^2 - t\rho_s > 0$, $u(t) = u_0 - t\rho_u \in \mathbb{C}$ such that $r(t)^2 s(t)^2 - |u(t)|^2 > 0$.

5.3 Geometric formality according to Kotshick

In this section we state the main theorem of this note, regarding class VII surfaces of the Enriques-Kodaira classification of compact complex surfaces.

Theorem 5.3.1 ([15]). *On class VII surfaces of the Enriques-Kodaira classification, geometric formality according to Kotshick is preserved by the Chern-Ricci flow starting at initial invariant Hermitian metrics.*

Proof. Let X be a class VII surface, that is, $\mathrm{Kod}(X) = -\infty$ and $b_1(X) = 1$. By [87, Theorem 6], for a compact oriented Kotschick-geometrically formal 4-manifold X, the first Betti number satisfies $b_1(X) \in \{0, 1, 2, 4\}$. Since all class VII surfaces are non-Kähler, they must have odd first Betti number by [33, 90], that is, $b_1(X) = 1$. By [87, Theorem 9], the Euler characteristic of such manifolds vanishes, implying that $b_2(X) = 0$. Since the characterization result by [30, 83, 95, 148], class VII surfaces with $b_2(X) = 0$ are necessarily Hopf or Inoue surfaces, then we see that the only Kotschick-geometrically formal class VII surfaces can be Hopf and Inoue surfaces. Therefore, Chern-Ricci flow starting at any metric cannot produce geometrically formal metrics on class VII surfaces other than Hopf and Inoue surfaces: we will then check the statement for those surfaces.

Case 1: Hopf surfaces

Hopf surfaces X are compact complex surfaces in class VII defined as a quotient of $\mathbb{C}^2 \smallsetminus \{0\}$ by a free action of a discrete group generated by a holomorphic contraction $\gamma(z, w) = (\alpha z + \lambda w^p, \beta w)$ where $\alpha, \beta, \lambda \in \mathbb{C}$ and $p \in \mathbb{N}$ are such that $0 < |\alpha| \le |\beta| < 1$ and $(\alpha - \beta^p)\lambda = 0$, see [91], [137, page 820], see [155, Remark 1].

The diffeomorphism type is $\mathbb{S}^1 \times \mathrm{SU}(2)$, and the complex structure is a special case of the Calabi-Eckmann complex structure on product of spheres [35]. See also [115, Theorem 4.1]. In terms of a coframe (φ^1, φ^2) of $(1,0)$-forms, they are described as

$$d\varphi^1 = \sqrt{-1}\varphi^1 \wedge \varphi^2 + \sqrt{-1}\varphi^1 \wedge \bar{\varphi}^2, \qquad d\varphi^2 = -\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1.$$

The de Rham cohomology of Hopf surfaces is

$$H^\bullet_{dR}(X; \mathbb{C}) \quad = \quad \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $2\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

We look at how harmonic representatives of de Rham cohomology change with respect to the invariant Hermitian metric, and in particular whether their product is still harmonic.

We notice that, varying the invariant Hermitian metric, the harmonic representatives are

$$H^\bullet_{dR}(X; \mathbb{R}) = \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\left\langle -\frac{1}{2} r^2 \varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2} u \varphi^{12\bar{2}} + \frac{1}{2} r^2 \varphi^{1\bar{1}\bar{2}} + \frac{\sqrt{-1}}{2} \bar{u} \varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle.$$

Indeed, it suffices to check that the harmonic representative of the class $[\varphi^2 - \bar\varphi^2]$ does not depend on the invariant metric. This is because harmonic representatives are invariant, and the class $[\varphi^2 - \bar\varphi^2] = \{\varphi^2 - \bar\varphi^2 + dc \ : \ c \in \mathbb{R}\}$ contains only one invariant representantive, which is then harmonic with respect to any metric. Then we compute the harmonic representative of the dual class in $H^3_{dR}(X;\mathbb{R})$ by applying the Hodge-star-operator to $\varphi^2 - \bar\varphi^2$ with respect to the arbitrary Hermitian metric. In any case, the product of an invariant 1-form and an invariant 3-form is either zero or a scalar multiple of the volume form. It follows that *any invariant metric on the Hopf surface is geometrically formal in the sense of Kotschick.*

Clearly, on the Hopf surface with invariant Hermitian metrics, the properties of geometric formality in the sense of Kotschick is preserved along the Chern-Ricci flow. Nonetheless, for completeness and for later use, we compute the Chern-Ricci form and the Chern-Ricci flow on X.

We start by computing the Chern-Riemann curvature of an invariant Hermitian metric. We follow notation as in [112, Section 2] (see also [96, Section 6] for another argument). With respect to the frame $(\varphi_1, \varphi_2, \bar\varphi_1, \bar\varphi_2)$ and to the dual coframe $\left(\varphi^1, \varphi^2, \bar\varphi^1, \bar\varphi^2\right)$, we set the structure constants

$$[\varphi_I, \varphi_H] =: c^K_{IH}\varphi_K.$$

Here, capital letters vary in $\{1, 2, \bar1, \bar2\}$, and the Einstein summation is assumed. In our case, the non-trivial structure constants are

$$\begin{array}{cccc}
c^1_{12} = -\sqrt{-1}, & c^2_{1\bar1} = \sqrt{-1}, & c^{\bar2}_{1\bar1} = \sqrt{-1}, & c^1_{1\bar2} = -\sqrt{-1}, \\
c^1_{\bar21} = \sqrt{-1}, & c^{\bar1}_{2\bar1} = -\sqrt{-1}, & c^2_{\bar11} = -\sqrt{-1}, & c^2_{\bar11} = -\sqrt{-1}, \\
c^{\bar1}_{\bar12} = \sqrt{-1}, & c^1_{\bar1\bar2} = \sqrt{-1}, & c^1_{\bar21} = \sqrt{-1}, & c^1_{\bar2\bar1} = -\sqrt{-1}.
\end{array}$$

Recall that the Christoffel symbols of the Levi-Civita connections (with respect to the above non-commutative frame) can be computed as

$$\begin{aligned}
(\Gamma^{LC})^K_{IH} &= \frac{1}{2} g^{KL}\left(g([\varphi_I, \varphi_H], \varphi_L) - g([\varphi_H, \varphi_L], \varphi_I) - g([\varphi_I, \varphi_L], \varphi_H)\right) \\
&= \frac{1}{2} c^K_{IH} - \frac{1}{2} g^{KA} g_{BI} c^B_{HA} - \frac{1}{2} g^{KA} g_{BH} c^B_{IA}.
\end{aligned}$$

Set $V = r^2 s^2 - |u|^2$ for simplicity. In our case, up to conjugation, the non-trivial ones are

$$\begin{gathered}
(\Gamma^{LC})^1_{11} = -s^2 u V^{-1}, \\
(\Gamma^{LC})^1_{12} = \tfrac{1}{2}(-\sqrt{-1}s^4 + \sqrt{-1}|u|^2)V^{-1}, \\
(\Gamma^{LC})^1_{1\bar1} = \tfrac{1}{2} s^2 \bar u V^{-1}, \\
(\Gamma^{LC})^1_{1\bar1} = \tfrac{1}{2} s^2 u V^{-1}, \\
(\Gamma^{LC})^1_{1\bar2} = -\tfrac{1}{2}\sqrt{-1}s^4 V^{-1}, \\
(\Gamma^{LC})^1_{1\bar2} = \tfrac{1}{2}\sqrt{-1}u^2 V^{-1}, \\
(\Gamma^{LC})^1_{21} = \tfrac{1}{2}(2\sqrt{-1}r^2 s^2 - \sqrt{-1}s^4 - \sqrt{-1}|u|^2)V^{-1}, \\
(\Gamma^{LC})^1_{22} = -s^2 \bar u V^{-1}, \\
(\Gamma^{LC})^1_{2\bar1} = -\tfrac{1}{2}\sqrt{-1}\bar u^2 V^{-1}, \\
(\Gamma^{LC})^{\bar1}_{2\bar1} = \tfrac{1}{2}(-2 * \sqrt{-1}r^2 s^2 + \sqrt{-1}s^4 + 2\sqrt{-1}|u|^2)V^{-1}, \\
(\Gamma^{LC})^1_{2\bar2} = -\tfrac{1}{2} s^2 \bar u V^{-1}, \\
(\Gamma^{LC})^1_{2\bar2} = -\tfrac{1}{2} s^2 u V^{-1},
\end{gathered}
\qquad
\begin{gathered}
(\Gamma^{LC})^2_{11} = -\sqrt{-1}u^2 V^{-1}, \\
(\Gamma^{LC})^2_{12} = -\tfrac{1}{2}(r^2 - s^2)u V^{-1}, \\
(\Gamma^{LC})^2_{1\bar1} = \tfrac{1}{2}\sqrt{-1}r^2 s^2 V^{-1}, \\
(\Gamma^{LC})^{\bar2}_{1\bar1} = \tfrac{1}{2}(\sqrt{-1}r^2 s^2 - 2\sqrt{-1}|u|^2)V^{-1}, \\
(\Gamma^{LC})^2_{1\bar2} = \tfrac{1}{2} s^2 u V^{-1}, \\
(\Gamma^{LC})^2_{1\bar2} = \tfrac{1}{2} r^2 u V^{-1}, \\
(\Gamma^{LC})^2_{21} = -\tfrac{1}{2}(r^2 - s^2)u V^{-1}, \\
(\Gamma^{LC})^2_{22} = -\sqrt{-1}|u|^2 V^{-1}, \\
(\Gamma^{LC})^2_{2\bar1} = \tfrac{1}{2} r^2 \bar u V^{-1}, \\
(\Gamma^{LC})^2_{2\bar1} = \tfrac{1}{2} s^2 \bar u V^{-1}, \\
(\Gamma^{LC})^2_{2\bar2} = -\tfrac{1}{2}\sqrt{-1}|u|^2 V^{-1}, \\
(\Gamma^{LC})^2_{2\bar2} = \tfrac{1}{2}\sqrt{-1}|u|^2 V^{-1}.
\end{gathered}$$

We can now compute the Christoffel symbols $(\Gamma^{Ch})^K_{IH}$ of the Chern connection by the formula [112, Equation (7)]:

$$(\Gamma^{\varepsilon,\rho})^K_{IH} = (\Gamma^{LC})^K_{IH} + \varepsilon g^{KL} T_{IHL} + \rho g^{KL} C_{IHL},$$

by setting $(\varepsilon,\rho) = \left(0, \frac{1}{2}\right)$, where

$$T_{IHL} = -d\omega(J\varphi_I, J\varphi_H, J\varphi_L), \qquad C_{IHL} = d\omega(J\varphi_I, \varphi_H, \varphi_L).$$

We get

$$\begin{array}{cc}
(\Gamma^{Ch})^2_{21} = -r^2 u V^{-1}, & (\Gamma^{Ch})^1_{21} = \sqrt{-1} r^2 s^2 V^{-1}, \\
(\Gamma^{Ch})^{\bar 2}_{1\bar 1} = \sqrt{-1}, & (\Gamma^{Ch})^{\bar 1}_{2\bar 1} = -\sqrt{-1}, \\
(\Gamma^{Ch})^1_{12} = -\sqrt{-1} s^4 V^{-1}, & (\Gamma^{Ch})^2_{12} = s^2 u V^{-1},
\end{array}$$

the others being equal to the corresponding Levi-Civita symbols or deduced by conjugation. We can compute the $(4,0)$-Riemannian curvature of $\nabla^{\varepsilon,\rho}$ as

$$\begin{aligned}
(R^{\varepsilon,\rho})_{IHKL} &= g_{AL}(\Gamma^{\varepsilon,\rho})^B_{HK}(\Gamma^{\varepsilon,\rho})^A_{IB} - g_{AL}(\Gamma^{\varepsilon,\rho})^B_{IK}(\Gamma^{\varepsilon,\rho})^A_{HB} \\
&\quad - g_{AL} c^B_{IH}(\Gamma^{\varepsilon,\rho})^A_{BK}.
\end{aligned}$$

By using the symmetries for the Chern curvature $(R^{Ch})_{IHKL} = -(R^{Ch})_{HIKL} = -(R^{Ch})_{IHLK}$ and the conjugation, we get that the only non-zero components are

$$\begin{array}{c}
(R^{Ch})_{1\bar 1 1\bar 1} = \frac{1}{2}(2r^4 s^2 - r^2 s^4 - 2(r^2 - s^2)|u|^2)V^{-1}, \\
(R^{Ch})_{1\bar 1 1\bar 2} = \frac{1}{2}(\sqrt{-1}|u|^2 u + (-\sqrt{-1} r^2 s^2 - \sqrt{-1} s^4)u)V^{-1}, \\
(R^{Ch})_{1\bar 1 2\bar 1} = \frac{1}{2}(-\sqrt{-1}|u|^2 \bar u - (-\sqrt{-1} r^2 s^2 - \sqrt{-1} s^4)\bar u)V^{-1}, \\
(R^{Ch})_{1\bar 1 2\bar 2} = \frac{1}{2} s^6 V^{-1}, \\
(R^{Ch})_{1\bar 2 1\bar 1} = \frac{1}{2}(-\sqrt{-1} r^2 s^2 u + 2\sqrt{-1}|u|^2 u)V^{-1}, \\
(R^{Ch})_{1\bar 2 1\bar 2} = \frac{1}{2} s^2 u^2 V^{-1}, \\
(R^{Ch})_{1\bar 1 2\bar 1} = -\frac{1}{2} s^2 |u|^2 V^{-1}, \\
(R^{Ch})_{(1\bar 2 2\bar 2} = \frac{1}{2}\sqrt{-1} s^4 u V^{-1}, \\
(R^{Ch})_{2\bar 1 1\bar 1} = \frac{1}{2}(\sqrt{-1} r^2 s^2 \bar u - 2\sqrt{-1}|u|^2 \bar u)V^{-1}, \\
(R^{Ch})_{2\bar 1 1\bar 2} = -\frac{1}{2} s^2 |u|^2 V^{-1}, \\
(R^{Ch})_{2\bar 1 2\bar 1} = \frac{1}{2} s^2 \bar u^2 V^{-1}, \\
(R^{Ch})_{2\bar 1 2\bar 2} = -\frac{1}{2}\sqrt{-1} s^4 \bar u V^{-1}, \\
(R^{Ch})_{2\bar 2 1\bar 1} = \frac{1}{2} r^2 |u|^2 V^{-1}, \\
(R^{Ch})_{2\bar 2 1\bar 2} = -\frac{1}{2}\sqrt{-1} r^2 s^2 u V^{-1}, \\
(R^{Ch})_{2\bar 2 2\bar 1} = \frac{1}{2}\sqrt{-1} r^2 s^2 \bar u V^{-1}, \\
(R^{Ch})_{2\bar 2 2\bar 2} = \frac{1}{2} s^2 |u|^2 V^{-1}.
\end{array}$$

Finally, we can compute the (first) Chern-Ricci curvature by tracing on the third and fourth indices:

$$(\mathrm{Ric}^{Ch})_{IH} = g^{KL}(R^{Ch})_{IHKL};$$

then the Cher-Ricci form can be defined as

$$\mathrm{Ric}^{Ch} = (\mathrm{Ric}^{Ch})_{ih}\sqrt{-1} dz^i \wedge d\bar z^h.$$

In our case, the only non-trivial coefficients are

$$(\mathrm{Ric}^{Ch})_{1\bar 1} = 2$$

and the corresponding $(\mathrm{Ric}^{Ch})_{\bar 1 1} = -(\mathrm{Ric}^{Ch})_{1\bar 1}$. Therefore the Chern-Ricci form of any invariant Hermitian metric is

$$\mathrm{Ric}^{Ch}(\omega) = 2\sqrt{-1}\varphi^1 \wedge \bar\varphi^1.$$

Therefore the solution of the Chern-Ricci flow starting at ω_0 of the form (5.2.7) is

$$2\omega(t) = \sqrt{-1}(r_0^2 - t)\varphi^{1\bar 1} + \sqrt{-1} s_0^2 \varphi^{2\bar 2} + u_0 \varphi^{1\bar 2} - \bar u_0 \varphi^{2\bar 1}, \tag{5.3.1}$$

defined as long as $t < \frac{r_0^2 s_0^2 - |u_0|^2}{s_0^2} < r_0^2$.

Case 2: Inoue surfaces

Inoue-Bombieri surfaces [76, 31] X are compact complex surfaces in class VII with second Betti number equal to zero and with no holomorphic curves [29, 30, 94, 95, 148]. Their universal cover is $\mathbb{C} \times \mathbb{H}$, where $\mathbb{H}$ denotes the upper half-plane. They are divided into three families, S_M, $S^+_{N,p,q,r;\mathbf{t}}$, and $S^-_{N,p,q,r}$, depending on parameters.

Case 2.1: Inoue-Bombieri surface of type S_M

We focus now on the case S_M: it has a structure of fibre bundle over $\mathbb{S}^1$, where the fibre is a 3-dimensional torus.

Inoue-Bombieri surfaces of type S_M admit a description as quotients of solvable Lie groups wih invariant complex structure [68], that we now describe. We can fix a coframe (φ^1, φ^2) of $(1,0)$-forms with structure equations

$$d\varphi^1 = \frac{\alpha - \sqrt{-1}\beta}{2\sqrt{-1}} \varphi^1 \wedge \varphi^2 - \frac{\alpha - \sqrt{-1}\beta}{2\sqrt{-1}} \varphi^1 \wedge \bar{\varphi}^2,$$

$$d\varphi^2 = -\sqrt{-1}\alpha \varphi^2 \wedge \bar{\varphi}^2,$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$. The de Rham cohomology of X can be explicitly described [149], see [12, Theorem 4.1]:

$$H^\bullet_{dR}(X; \mathbb{R}) \quad = \quad \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $2\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (5.2.1):

$$H^\bullet_{dR}(X; \mathbb{R}) = \qquad (5.3.2)$$
$$\mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\left\langle -\frac{1}{2} r^2 \varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2} u \varphi^{12\bar{2}} + \frac{1}{2} r^2 \varphi^{1\bar{1}\bar{2}} + \frac{\sqrt{-1}}{2} \bar{u} \varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle$$

We conclude that: *any invariant Hermitian metric on an Inoue surface of type S_M is geometrically formal in the sense of Kotschick.*

The Chern-Ricci form of any invariant Hermitian metric is

$$2\mathrm{Ric}^{Ch}(\omega) = -\sqrt{-1}\alpha^2 \varphi^2 \wedge \bar{\varphi}^2,$$

whence the solution of the Chern-Ricci flow (CRF) is given by

$$2\omega(t) = \sqrt{-1} r_0^2 \varphi^{1\bar{1}} + \sqrt{-1}\left(s_0^2 + \alpha^2 t\right) \varphi^{2\bar{2}} + u_0 \varphi^{1\bar{2}} - \bar{u}_0 \varphi^{2\bar{1}}, \qquad (5.3.3)$$

defined for any non-negative time $t \geq 0$.

Clearly, on an Inoue surface of type S_M with invariant Hermitian metrics, the properties of geometric formality in the sense of Kotschick is preserved along the Chern-Ricci flow.

Case 2.2: Inoue surfaces of class $S^\pm$

In this subsection, we focus on the case of Inoue surfaces of type $S^\pm$. Inoue-Bombieri surfaces of type S^- have an unramified double cover of type S^+: we can then restrict to Inoue-Bombieri surfaces of type S^+, which have a structure of fibre bundle over $\mathbb{S}^1$, where the fibre is a compact quotient of the 3-dimensional Heisenberg group.

Also Inoue-Bombieri surfaces of type S^+ admit a description as quotients of solvable Lie groups [68], that we now describe. We can fix a coframe (φ^1, φ^2) of $(1,0)$-forms with structure equations

$$d\varphi^1 = \frac{1}{2\sqrt{-1}}\varphi^1 \wedge \varphi^2 + \frac{1}{2\sqrt{-1}}\varphi^2 \wedge \bar{\varphi}^1 + \frac{q\sqrt{-1}}{2}\varphi^2 \wedge \bar{\varphi}^2,$$

$$d\varphi^2 = \frac{1}{2\sqrt{-1}}\varphi^2 \wedge \bar{\varphi}^2,$$

where $q \in \mathbb{R}$. The de Rham cohomology of X can be explicitly described [149], see [12, Theorem 4.1]:

$$H^\bullet_{dR}(X;\mathbb{C}) \quad = \quad \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^2 - \varphi^{\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

The situation is exactly as in (5.3.2). We conclude that: *any invariant Hermitian metric on an Inoue surface of type $S^\pm$ is geometrically formal in the sense of Kotschick.*

The Chern-Ricci form of any invariant Hermitian metric is

$$2\mathrm{Ric}^{Ch}(\omega) = -\sqrt{-1}\varphi^{2\bar{2}},$$

whence the solution of the Chern-Ricci flow (CRF) is given by

$$2\omega(t) = \sqrt{-1}r_0^2\varphi^{1\bar{1}} + \sqrt{-1}\left(s_0^2 + t\right)\varphi^{2\bar{2}} + u_0\varphi^{1\bar{2}} - \bar{u}_0\varphi^{2\bar{1}}, \tag{5.3.4}$$

defined for any non-negative time $t \geq 0$.

Clearly, on an Inoue surface of type $S^\pm$ with invariant Hermitian metrics, the properties of geometric formality in the sense of Kotschick is preserved along the Chern-Ricci flow. □

We also analyze in details primary and secondary Kodaira surfaces resulting in the following proposition, for which we give explicit computations.

Proposition 5.3.2 ([15]). *On any Kodaira surface, the properties of geometric formality in the sense of Kotschick is preserved along the Chern-Ricci flow starting at initial invariant Hermitian metrics.*

Proof. We will look at each case separately.

Case 1: Primary Kodaira surface

Kodaira surfaces X are compact complex surfaces of Kodaira dimension $\mathrm{Kod}(X) = 0$ and first Betti number $b_1(X) = 3$. Primary Kodaira surfaces have trivial canonical bundle.

We note that, by [87, Theorem 6], primary Kodaira surfaces are never Kotschick-geometrically formal, not even with regards to non-invariant metrics, by having $b_1 = 3$: hence *Chern-Ricci flow preserves geometric formality according to Kotschick*. The same conclusion follows by [69, Theorem 1] stating that non-tori nilmanifolds are never formal, therefore never geometrically formal in the sense of Kotschick. Nevertheless we give explicit computations for this fact.

We recall the description of primary Kodaira surfaces as quotients of solvable Lie groups [68]. There exists a coframe (φ^1, φ^2) of $(1,0)$-forms with structure equations

$$d\varphi^1 = 0, \qquad d\varphi^2 = \frac{\sqrt{-1}}{2}\varphi^1 \wedge \bar{\varphi}^1.$$

The de Rham cohomology of X can be explicitly described:

$$\begin{aligned} H^{\bullet}_{dR}(X;\mathbb{R}) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^1, \varphi^{\bar 1}, \varphi^2 - \varphi^{\bar 2}\rangle \oplus \mathbb{C}\langle \varphi^{12}, \varphi^{1\bar 2}, \varphi^{2\bar 1}, \varphi^{\bar 1\bar 2}\rangle \\ &\oplus \mathbb{C}\langle \varphi^{12\bar 2}, \varphi^{2\bar 1\bar 2}, \varphi^{12\bar 1} - \varphi^{1\bar 1\bar 2}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1\bar 2}\rangle \end{aligned}$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar 1} + \sqrt{-1}\varphi^{2\bar 2}$ instead of their classes.

We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (5.2.1):

$$\begin{aligned} H^{\bullet}_{dR}(X;\mathbb{R}) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^1, \varphi^{\bar 1}, \varphi^2 - \varphi^{\bar 2}\rangle \oplus \mathbb{C}\left\langle \varphi^{12}, \varphi^{1\bar 2} + \frac{\sqrt{-1}\,\bar u}{s^2}\varphi^{1\bar 1}, \varphi^{2\bar 1} - \frac{\sqrt{-1}\,u}{s^2}\varphi^{1\bar 1}, \varphi^{\bar 1\bar 2}\right\rangle \\ &\oplus \mathbb{C}\left\langle \frac{1}{2}s^2\varphi^{12\bar 2} + \frac{\sqrt{-1}}{2}\bar u\varphi^{12\bar 1}, \frac{1}{2}s^2\varphi^{2\bar 1\bar 2} - \frac{\sqrt{-1}}{2}u\varphi^{1\bar 1\bar 2}, \right. \\ &\left. -\frac{1}{2}r^2\varphi^{12\bar 1} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar 2} + \frac{1}{2}r^2\varphi^{1\bar 1\bar 2} + \frac{\sqrt{-1}}{2}\bar u\varphi^{2\bar 1\bar 2}\right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1\bar 2}\rangle, \end{aligned}$$

We explicitly notice that, *on primary Kodaira surfaces, an invariant Hermitian metric is never geometrically formal in the sense of Kotschick*: indeed, $\varphi^1 \wedge \varphi^{\bar 1\bar 2}$ is never harmonic.

As for Chern-Ricci flow, the primary Kodaira surface has trivial canonical bundle, therefore $\mathrm{Ric}^{Ch}(\omega) = 0$. Then, clearly, the Chern-Ricci flow does not evolve invariant Hermitian metrics.

Case 2: Secondary Kodaira surface

Secondary Kodaira surfaces X are quotients of primary Kodaira surfaces by finite groups; they have torsion non-trivial canonical bundle.

We recall the description of secondary Kodaira surfaces as quotients of solvable Lie groups [68]. There exists a coframe (φ^1, φ^2) of $(1,0)$-forms with structure equations

$$d\varphi^1 = -\frac{1}{2}\varphi^1 \wedge \varphi^2 + \frac{1}{2}\varphi^1 \wedge \bar\varphi^2, \qquad d\varphi^2 = \frac{\sqrt{-1}}{2}\varphi^1 \wedge \bar\varphi^1.$$

The cohomologies of X can be explicitly described:

$$H^{\bullet}_{dR}(X;\mathbb{R}) = \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^2 - \varphi^{\bar 2}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1} - \varphi^{1\bar 1\bar 2}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1\bar 2}\rangle,$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar 1} + \sqrt{-1}\varphi^{2\bar 2}$ instead of their classes.

As for the harmonic representatives for de Rham cohomology, the situation is very similar to the Inoue case. We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (5.2.1):

$$\begin{aligned} H^{\bullet}_{dR}(X;\mathbb{R}) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^2 - \varphi^{\bar 2}\rangle \\ &\oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{12\bar 1} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar 2} + \frac{1}{2}r^2\varphi^{1\bar 1\bar 2} + \frac{\sqrt{-1}}{2}\bar u\varphi^{2\bar 1\bar 2}\right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1\bar 2}\rangle. \end{aligned}$$

We conclude that *any invariant Hermitian metric on an secondary Kodaira is geometrically formal in the sense of Kotschick.*

As for the Chern-Ricci flow, the secondary Kodaira surface has torsion canonical bundle, therefore $\mathrm{Ric}^{Ch}(\omega) = 0$. Therefore, the Chern-Ricci flow does not evolve invariant Hermitian metrics. □

5.4 Dolbeault and Bott-Chern geometric formalities

As for Dolbeault or Bott-Chern geometric formality, the situation is clear for Hopf, Inoue and Kodaira surfaces, as we now describe. We also make computations regarding Aeppli cohomology and harmonic representatives with respect to the Aeppli Laplacian.

Proposition 5.4.1 ([15]). *On Hopf, Inoue, and Kodaira surfaces, the property of Dolbeault geometric formality and of Bott-Chern geometric formality is preserved along the Chern-Ricci flow starting at invariant metrics. In the same situation, the properties of having a structure of algebra or a structure of H_{BC}-module for harmonic-Aeppli forms are all preserved by the Chern Ricci flow.*

Proof. We refer to the complex structures used in Theorem 5.3.1 and Proposition 5.3.2, for the computation of Dolbeault, Bott-Chern, and Aeppli cohomologies.

Hopf surfaces

The Dolbeault cohomology of Hopf surfaces is explicitly described in [73, Appendix II, Theorem 9.5], and see [12, Section 3.1] for the Bott-Chern cohomology:

$$
\begin{aligned}
H^{\bullet,\bullet}_{\bar\partial}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{\bar 2}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1\bar 2}\rangle,\\
H^{\bullet,\bullet}_{BC}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{1\bar 1}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar 1\bar 2}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1\bar 2}\rangle,\\
H^{\bullet,\bullet}_{A}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{2}\rangle \oplus \mathbb{C}\langle \varphi^{\bar 2}\rangle \oplus \mathbb{C}\langle \varphi^{2\bar 2}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1\bar 2}\rangle,
\end{aligned}
$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $2\omega = \sqrt{-1}\varphi^{1\bar 1} + \sqrt{-1}\varphi^{2\bar 2}$ instead of their classes.

We look at how the harmonic representatives of such cohomologies change with respect to the invariant Hermitian metric, and in particular when their product is still harmonic.

We summarize them as follows:

$$
\begin{aligned}
H^{\bullet,\bullet}_{\bar\partial}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{\bar 2}\rangle \oplus \mathbb{C}\langle -\frac{1}{2}r^2\varphi^{1\bar 1\bar 2} - \frac{\sqrt{-1}}{2}\bar u\varphi^{2\bar 1\bar 2}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1\bar 2}\rangle,\\
H^{\bullet,\bullet}_{BC}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{1\bar 1}\rangle \oplus \mathbb{C}\langle -\frac{1}{2}r^2\varphi^{12\bar 1} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar 2}\rangle \oplus \mathbb{C}\langle -\frac{1}{2}r^2\varphi^{1\bar 1\bar 2} - \frac{\sqrt{-1}}{2}\bar u\varphi^{2\bar 1\bar 2}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1\bar 2}\rangle,\\
H^{\bullet,\bullet}_{A}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{2}\rangle \oplus \mathbb{C}\langle \varphi^{\bar 2}\rangle \oplus \mathbb{C}\langle s^4\varphi^{2\bar 2} + |u|^2\varphi^{1\bar 1} - \sqrt{-1}s^2\, u\varphi^{1\bar 2} + \sqrt{-1}s^2\bar u\varphi^{2\bar 1}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar 1\bar 2}\rangle,
\end{aligned}
$$

Let us focus first on the Dolbeault cohomology. Here, the only Dolbeault-harmonic representative that changes is for the generator in $H^{1,2}_{\bar\partial}(X)$. We conclude that *any invariant Hermitian metric on the Hopf surface is geometrically-Dolbeault formal.*

As regards the Bott-Chern cohomology, to our aim, that is, studying harmonicity of products of Bott-Chern-harmonic forms, the only case of interest is the product $[\varphi^{1\bar 1}] \smile [\varphi^{1\bar 1}]$, the products with the class $[1]$ being trivial and the other products being zero because of degree reasons. Since the harmonic representatives with respect to invariant metrics are invariant, the Bott-Chern class $[\varphi^{1\bar 1}] = \{\varphi^{1\bar 1} + \partial\bar\partial c \,:\, c \in \mathbb{R}\}$ contains only one invariant representantive, that is also harmonic with respect to any invariant Hermitian metric. Again, we have that *any invariant Hermitian metric on the Hopf surface is geometrically-Bott-Chern formal.*

We consider the Aeppli cohomology. On the one side, we can consider the products between Aeppli-harmonic forms: the only possibly non-trivial products concern the classes $[\varphi^2]$ and $[\bar\varphi^2]$, $[\varphi^2]$ and $[\varphi^2 \wedge \bar\varphi^2]$, $[\bar\varphi^2]$ and $[\varphi^2 \wedge \bar\varphi^2]$. Since the classes $[\varphi^2]$ and $[\bar\varphi^2]$ contain only one invariant representative, we are reduced to study how the harmonic representative of the Aeppli cohomology class $[\varphi^2 \wedge \bar{\varphi}^2]$ depends on the invariant Hermitian metric. The arbitrary representative in the Aeppli cohomology class $[\varphi^{2\bar{2}}]$ is

$$\begin{aligned} h &:= \varphi^2 \wedge \bar{\varphi}^2 + \partial\left(\lambda_1 \bar{\varphi}^1 + \lambda_2 \bar{\varphi}^2\right) + \overline{\partial}\left(\lambda_3 \varphi^1 + \lambda_4 \varphi^2\right) \\ &= \varphi^{2\bar{2}} - \sqrt{-1}\left(\lambda_2 + \lambda_4\right)\varphi^{1\bar{1}} + \sqrt{-1}\lambda_1 \varphi^{2\bar{1}} + \sqrt{-1}\lambda_3 \varphi^{1\bar{2}}, \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$. By (5.2.5), we compute

$$\begin{aligned} V \cdot *_g h &= V \cdot \left(*_g \varphi^{2\bar{2}} + \sqrt{-1}\lambda_1 *_g \varphi^{2\bar{1}} - \sqrt{-1}\left(\lambda_2 + \lambda_4\right) *_g \varphi^{1\bar{1}} + \sqrt{-1}\lambda_3 *_g \varphi^{1\bar{2}} \right) \\ &= \left(r^4 - \lambda_1 u r^2 - \sqrt{-1}(\lambda_2 + \lambda_4)|u|^2 + \lambda_3 \overline{u} r^2 \right) \varphi^{1\bar{1}} \\ &\quad + \left(-\sqrt{-1} u r^2 + \sqrt{-1}\lambda_1 u^2 - (\lambda_2 + \lambda_4) u s^2 - \sqrt{-1}\lambda_3 r^2 s^2 \right) \varphi^{1\bar{2}} \\ &\quad + \left(\sqrt{-1}\overline{u} r^2 - \sqrt{-1}\lambda_1 r^2 s^2 + (\lambda_2 + \lambda_4)\overline{u} s^2 + \sqrt{-1}\lambda_3 \overline{u}^2 \right) \varphi^{2\bar{1}} \\ &\quad + \left(|u|^2 - \lambda_1 u s^2 - \sqrt{-1}(\lambda_2 + \lambda_4) s^4 + \lambda_3 \overline{u} s^2 \right) \varphi^{2\bar{2}}. \end{aligned}$$

By using the structure equations, we now compute

$$\begin{aligned} \partial(*_g h) &= \sqrt{-1}\frac{-\sqrt{-1}ur^2 + \sqrt{-1}\lambda_1 u^2 - (\lambda_2 + \lambda_4)us^2 - \sqrt{-1}\lambda_3 r^2 s^2}{r^2 s^2 - |u|^2}\varphi^{12\bar{2}} \\ &\quad -\sqrt{-1}\frac{|u|^2 - \lambda_1 u s^2 - \sqrt{-1}(\lambda_2 + \lambda_4)s^4 + \lambda_3 \overline{u} s^2}{r^2 s^2 - |u|^2}\varphi^{12\bar{1}}, \\ \overline{\partial}(*_g h) &= \sqrt{-1}\frac{\sqrt{-1}\overline{u} r^2 - \sqrt{-1}\lambda_1 r^2 s^2 + (\lambda_2 + \lambda_4)\overline{u} s^2 + \sqrt{-1}\lambda_3 \overline{u}^2}{r^2 s^2 - |u|^2}\varphi^{2\bar{1}\bar{2}} \\ &\quad -\sqrt{-1}\frac{|u|^2 - \lambda_1 u s^2 - \sqrt{-1}(\lambda_2 + \lambda_4)s^4 + \lambda_3 \overline{u} s^2}{r^2 s^2 - |u|^2}\varphi^{1\bar{1}\bar{2}}. \end{aligned}$$

Therefore the Aeppli-harmonicity conditions $\partial\overline{\partial}h = \partial *_g h = \overline{\partial} *_g h = 0$ yield

$$\begin{pmatrix} \sqrt{-1}u^2 & -us^2 & -\sqrt{-1}r^2 s^2 & -us^2 \\ -us^2 & -\sqrt{-1}s^4 & \overline{u}s^2 & -\sqrt{-1}s^4 \\ -\sqrt{-1}r^2 s^2 & \overline{u}s^2 & \sqrt{-1}\overline{u}^2 & \overline{u}s^2 \\ -us^2 & -\sqrt{-1}s^4 & \overline{u}s^2 & -\sqrt{-1}s^4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} \sqrt{-1}ur^2 \\ -|u|^2 \\ -\sqrt{-1}\overline{u}r^2 \\ -|u|^2 \end{pmatrix},$$

where the rank of the first matrix is 3 thanks to the condition $r^2 s^2 - |u|^2 > 0$. By solving the system, we get

$$\begin{aligned} \lambda_1 &= \frac{\overline{u}}{s^2}, \\ \lambda_2 &= \frac{\sqrt{-1}|u|^2}{s^4} - \lambda, \\ \lambda_3 &= -\frac{u}{s^2}, \\ \lambda_4 &= \lambda, \end{aligned}$$

varying $\lambda \in \mathbb{C}$. We finally get that the harmonic representative of $[\varphi^{2\bar{2}}]$ with respect to g is

$$h = \frac{|u|^2}{s^4}\varphi^{1\bar{1}} - \frac{\sqrt{-1}u}{s^2}\varphi^{1\bar{2}} + \frac{\sqrt{-1}\overline{u}}{s^2}\varphi^{2\bar{1}} + \varphi^{2\bar{2}}.$$

At the end of the day, we get that: *Aeppli-hamornic forms have a structure of algebra if and only if the metric* (5.2.1) *is diagonal, namely,* $u = 0$.

Finally, we consider the Aeppli cohomology as a Bott-Chern-cohomology-module. By the Stokes theorem, there is no invariant exact 4-form other than the zero form; in particular, any invariant $(2,2)$-form is harmonic with respect to any Hermitian invariant metric. This reduces to consider only the products $[\varphi^{1\bar{1}}]_{BC} \smile [\varphi^2]_A$ and $[\varphi^{1\bar{1}}]_{BC} \smile [\bar{\varphi}^2]_A$. By the argument above, both $[\varphi^{1\bar{1}}]_{BC}$ and $[\varphi^2]_A$, respectively $[\bar{\varphi}^2]_A$, contain only one invariant representative that is harmonic with respect to any Hermitian metric. Therefore: *for any invariant Hermitian metric on the Hopf surface, Aeppli-harmonic forms have a structure of module over Bott-Chern-harmonic forms.*

As regards the Chern-Ricci flow, we already have an expression for it computed in (5.3.1). Clearly, then, on the Hopf surface with invariant Hermitian metrics, *the properties of geometric-Dolbeault formality, of geometric-Bott-Chern formality, of the Aeppli-harmonic forms having a structure of algebra, of the Aeppli-hamornic forms having a structure of module over Bott-Chern-harmonic forms, are all preserved along the Chern-Ricci flow.*

Inoue-Bombieri surfaces of type S_M

The cohomologies of Inoue-Bombieri surfaces of type S_M can be explicitly described [149], see [12, Theorem 4.1]:

$$\begin{aligned}
H^{\bullet,\bullet}_{\bar\partial}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle,\\
H^{\bullet,\bullet}_{BC}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{2\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle,\\
H^{\bullet,\bullet}_{A}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{2}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle,
\end{aligned}$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $2\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (5.2.1):

$$\begin{aligned}
H^{\bullet,\bullet}_{\bar\partial}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}}\rangle \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}}\right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle,\\
H^{\bullet,\bullet}_{BC}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{2\bar{2}}\rangle \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}}\right\rangle\\
&\quad \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar{2}}\right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle, \qquad (5.4.1)\\
H^{\bullet,\bullet}_{A}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{2}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}}\rangle\\
&\quad \oplus \mathbb{C}\left\langle \varphi^{1\bar{1}} - \frac{\sqrt{-1}\,u}{r^2}\varphi^{1\bar{2}} + \frac{\sqrt{-1}\,\bar{u}}{r^2}\varphi^{2\bar{1}} + \frac{|u|^2}{r^4}\varphi^{2\bar{2}}\right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle,
\end{aligned}$$

We conclude that: *any invariant Hermitian metric on an Inoue surface of type S_M is geometrically-Dolbeault formal, is geometrically-Bott-Chern formal, and the Aeppli-harmonic forms have a structure of module over Bott-Chern-harmonic forms. On the other hand, Aeppli-harmonic forms have a structure of algebra if and only if the metric is diagonal.*

The Chern-Ricci flow has expression as in (5.3.3). Clearly, we can state that on an Inoue surface of type S_M with invariant Hermitian metrics, *the properties of Dolbeault-geometric formality, of Bott-Chern-geometric formality, of the Aeppli-harmonic forms having a structure of algebra, of the Aeppli-hamornic forms having a structure of module over Bott-Chern-harmonic forms, are all preserved along the Chern-Ricci flow.*

Inoue surfaces of type $S^{\pm}$

The cohomologies of Inoue surfaces of type $S^{\pm}$ can be explicitly described [149], see [12, Theorem 4.1]:

$$\begin{aligned} H^{\bullet,\bullet}_{\overline{\partial}}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle, \\ H^{\bullet,\bullet}_{BC}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{2\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle, \\ H^{\bullet,\bullet}_{A}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{2}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle, \end{aligned}$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

As for the harmonic representatives of Dolbeault, Bott-Chern and Aeppli cohomologies, the situation is exactly as (5.4.1).

We conclude that: *any invariant Hermitian metric on an Inoue surface of type $S^{\pm}$ is geometrically-Dolbeault formal, is geometrically-Bott-Chern formal, and the Aeppli-harmonic forms have a structure of module over Bott-Chern-harmonic forms. On the other hand, Aeppli-harmonic forms have a structure of algebra if and only if the metric is diagonal.*

The Chern-Ricci flow has expression as in (5.3.4). Hence, we have that on an Inoue surface of type $S^{\pm}$ with invariant Hermitian metrics, *the properties of geometric-Dolbeault formality, of geometric-Bott-Chern formality, of the Aeppli-harmonic forms having a structure of algebra, of the Aeppli-hamornic forms having a structure of module over Bott-Chern-harmonic forms, are all preserved along the Chern-Ricci flow.*

Primary Kodaira surfaces

The cohomologies of primary Kodaira surfaces can be explicitly described [149], see [12, Theorem 4.1]:

$$\begin{aligned} H^{\bullet,\bullet}_{\overline{\partial}}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{1}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}},\varphi^{\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}},\varphi^{2\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}}\rangle \\ &\quad \oplus \mathbb{C}\langle \varphi^{12\bar{1}},\varphi^{12\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{2\bar{1}\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle, \\ H^{\bullet,\bullet}_{BC}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{1}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{12}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}},\varphi^{1\bar{2}},\varphi^{2\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}}\rangle \\ &\quad \oplus \mathbb{C}\langle \varphi^{12\bar{1}},\varphi^{12\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}\bar{2}},\varphi^{2\bar{1}\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle, \\ H^{\bullet,\bullet}_{A}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{1},\varphi^{2}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}},\varphi^{\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}},\varphi^{2\bar{1}},\varphi^{2\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}}\rangle \\ &\quad \oplus \mathbb{C}\langle \varphi^{12\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{2\bar{1}\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle, \end{aligned}$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (5.2.1):

$$\begin{aligned} H^{\bullet,\bullet}_{\overline{\partial}}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{1}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}},\varphi^{\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}} - \sqrt{-1}\,s\varphi^{1\bar{1}},\varphi^{2\bar{1}} + \sqrt{-1}\,s\varphi^{1\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}}\rangle \\ &\quad \oplus \mathbb{C}\,\langle -\frac{1}{2}\,r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}\,u\varphi^{12\bar{2}}, \frac{1}{2}\,s^2\varphi^{12\bar{2}} + \frac{\sqrt{-1}}{2}\,\bar{u}\varphi^{12\bar{1}}\rangle \\ &\quad \oplus \mathbb{C}\,\langle \frac{1}{2}\,s^2\varphi^{2\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\,u\varphi^{1\bar{1}\bar{2}}\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}}\rangle, \\ H^{\bullet,\bullet}_{BC}(X) &= \mathbb{C}\langle 1\rangle \oplus \mathbb{C}\langle \varphi^{1}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{12}\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}},\varphi^{1\bar{2}},\varphi^{2\bar{1}}\rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}}\rangle \\ &\quad \oplus \mathbb{C}\,\langle -\frac{1}{2}\,r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}\,u\varphi^{12\bar{2}}, \frac{\sqrt{-1}}{2}\,\bar{u}\varphi^{12\bar{1}} + \frac{1}{2}\,s^2\varphi^{12\bar{2}}\rangle \end{aligned}$$

$$
\begin{aligned}
& \oplus \mathbb{C}\left\langle \frac{1}{2} s^2 \varphi^{2\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2} u\varphi^{1\bar{1}\bar{2}}, -\frac{1}{2} r^2 \varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2} \bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\
H_A^{\bullet,\bullet}(X) \quad = \quad & \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^1, \varphi^2 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12} \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}} \rangle \\
& \oplus \mathbb{C}\left\langle s^2 \varphi^{1\bar{2}} + \sqrt{-1}\,\bar{u}\,\varphi^{1\bar{1}},\ s^2 \varphi^{2\bar{1}} - \sqrt{-1}\,u\,\varphi^{1\bar{1}},\ s^4 \varphi^{2\bar{2}} - |u|^2 \varphi^{1\bar{1}} \right\rangle \\
& \oplus \mathbb{C}\left\langle \frac{1}{2} s^2 \varphi^{12\bar{2}} + \frac{\sqrt{-1}}{2} \bar{u}\varphi^{12\bar{1}} \right\rangle \oplus \mathbb{C}\left\langle \frac{1}{2} s^2 \varphi^{2\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2} u\varphi^{1\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle,
\end{aligned}
$$

We notice that for primary Kodaira surfaces *an invariant Hermitian metric is never geometrically-Dolbeault formal, e.g.* $\varphi^1 \wedge \bar{\varphi}^1$ is never Dolbeault-harmonic. In fact, Cattaneo and Tomassini noticed in [37, Example 4.3] that primary Kodaira surfaces have a non-vanishing Dolbeault-Massey triple product, whence they are not Dolbeault formal in the sense of [106]. Also, it is *never geometrically-Bott-Chern formal, e.g.* $\varphi^1 \wedge \varphi^{\bar{1}\bar{2}}$ is never Bott-Chern-harmonic. *The space of Aeppli-harmonic forms is never an algebra, e.g.* $\varphi^1 \wedge \bar{\varphi}^1$ is never Aeppli-harmonic, *neither a module over the space of Bott-Chern harmonic forms, e.g.* $\varphi^1 \wedge \bar{\varphi}^1$ is never Aeppli-harmonic.

The primary Kodaira surface has trivial canonical bundle, therefore $\mathrm{Ric}^{Ch}(\omega) = 0$. Then the Chern-Ricci flow does not evolve invariant Hermitian metrics.

Then clearly on a primary Kodaira surface with invariant Hermitian metrics, *the properties of geometric-Dolbeault formality, of geometric-Bott-Chern formality, of the Aeppli-harmonic forms having a structure of algebra, of the Aeppli-hamornic forms having a structure of module over Bott-Chern-harmonic forms, are all preserved along the Chern-Ricci flow.*

Secondary Kodaira surfaces

The cohomologies of secondary Kodaira surfaces can be explicitly described [149], see [12, Theorem 4.1]:

$$
\begin{aligned}
H_{\bar{\partial}}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\
H_{BC}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\
H_A^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{2\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle,
\end{aligned}
$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

As for the harmonic representatives for Dolbeault, Bott-Chern and Aeppli cohomologies, the situation is very similar to the Inoue case, only the computations for the class $[\varphi^{2\bar{2}}] \in H_A^{1,1}(X)$ being slightly different.

We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (5.2.1):

$$
\begin{aligned}
H_{\bar{\partial}}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\left\langle -\frac{1}{2} r^2 \varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2} \bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle \\
H_{BC}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}} \rangle \oplus \mathbb{C}\left\langle -\frac{1}{2} r^2 \varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2} \bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \\
&\quad \oplus \mathbb{C}\left\langle -\frac{1}{2} r^2 \varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2} u\varphi^{12\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\
H_A^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \\
&\quad \oplus \mathbb{C}\left\langle |u|^2 \varphi^{1\bar{1}} - \sqrt{-1}\, s^2 u\varphi^{1\bar{2}} + \sqrt{-1}\, s^2 \bar{u}\varphi^{2\bar{1}} + s^4 \varphi^{2\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle.
\end{aligned}
$$

We conclude that: *any invariant Hermitian metric on a secondary Kodaira surface is geometrically-Dolbeault formal, is geometrically-Bott-Chern formal, and the Aeppli-harmonic forms* *have a structure of module over Bott-Chern-harmonic forms. On the other hand, Aeppli-harmonic forms have a structure of algebra if and only if the metric is diagonal.*

The secondary Kodaira surface has torsion canonical bundle, therefore $\mathrm{Ric}^{Ch}(\omega) = 0$. Then the Chern-Ricci flow does not evolve invariant Hermitian metrics.

Then clearly on a secondary Kodaira surface with invariant Hermitian metrics, *the properties of geometric-Dolbeault formality, of geometric-Bott-Chern formality, of the Aeppli-harmonic forms having a structure of algebra, of the Aeppli-hamornic forms having a structure of module over Bott-Chern-harmonic forms, are all preserved along the Chern-Ricci flow.* □

We summarize the results in the last two Sections in Table 5.1.

surface	**Kotschick geom. form.**	**Dolbeault geom. form.**	**Bott-Chern geom. form.**	**Aeppli harm. f. as algebra**	**Aeppli harm. f. as BC-module**
class $\mathrm{VII}^{b_2>0}$	never	?	?	?	?
Hopf (invariant metrics)	always	always	always	diagonal	always
Inoue-Bombieri S_M (invariant metrics)	always	always	always	diagonal	always
Inoue $S^{\pm}$ (invariant metrics)	always	always	always	diagonal	always
primary Kodaira	never	never	never	never	never
secondary Kodaira (invariant metrics)	always	always	always	diagonal	always

Table 5.1: Summary of Theorem 5.3.1 and Propositions 5.3.2 and 5.4.1 concerning geometric formalities (for Kotschick, Dolbeault, Bott-Chern) and the structure of Aeppli-harmonic forms with respect to Hermitian metrics, respectively invariant Hermitian metrics on Hopf, Inoue, Kodaira surfaces.

In view of further study, we notice that:

- in any mentioned cases, the Chern-Ricci flow starting at an invariant metric clearly preserves each one of the above properties, since it preserves diagonal metrics. (Compare also [88, Proposition 3], showing that, for certain G-homogeneous spaces, every G-invariant metrics is geometrically formal.) We ask whether this behaviour is more general, or whether there exists a counterexample for which the Chern-Ricci flow does *not* preserve the geometric formality in the sense of Kotschick, or any other of the geometric Hermitian formalities discussed above. We notice that the above invariant metrics are Gauduchon, that is pluriclosed (also known as SKT) being defined on four-dimensional manifolds. Therefore, as the Referee kindly suggested to us, it may be interesting to further investigate the 6-dimensional nilmanifolds admitting invariant SKT metrics as classified in [53].

- Clearly, holomorphically-parallelizable manifolds do not provide such counterexamples when restricting to invariant metrics, since they have holomorphically-trivial canonical bundle, whence invariant Hermitian metrics are Chern-Ricci-flat. Our attempts on four-dimensional Lie groups (possibly not admitting compact quotients), as in [113] and references therein, or small deformations of the Iwasawa manifold [105, 11] still have not provided further examples.

- The same question may be addressed for other geometric flows other than the Chern-Ricci flow, for example the Hermitian curvature flows in [140] or in particular the one studied in [153].

- It could be interesting to further investigate Massey triple products and Dolbeault Massey products, see [150, 37], or other Massey products, in particular on class VII surfaces with $b_2 > 0$ and on primary Kodaira surfaces.

Chapter 6

Cohomological and formal properties of Strong Kähler with torsion and astheno-Kähler metrics

In this chapter, we first construct a family of simply-connected 2-step nilpotent Lie groups G, admitting discrete uniform subgroups Γ and endowed with a left-invariant complex structure J, such that $(\Gamma\backslash G, J)$ carries an astheno-Kähler metric (see Theorem 6.2.1 for the precise statement). Such a construction will be applied in the study of the behaviour of blow-ups. In fact, in [54] respectively [55, Proposition 2.4] it is proved that the existence of an SKT metric respectively a Hermitian metric g with fundamental form F on an n-dimensional compact complex manifold M, satisfying $\partial\overline{\partial}F = 0$, $\partial\overline{\partial}F^2 = 0$, is stable under blow-ups of M.

In contrast, in Theorem 6.3.3 we prove that the existence of a Hermitian metric g with fundamental form F satisfying $\partial\overline{\partial}F^{n-2} = 0, \quad \partial\overline{\partial}F^{n-3} = 0$ but $\partial\overline{\partial}F \neq 0$, is not preserved by blow-ups.

Then, we investigate the relation between SKT metrics and geometrically-Bott-Chern-formal metrics. More precisely, we study the 6-dimensional nilmanifolds with a left-invariant complex structure admitting a left-invariant SKT metric, which have been characterized by Fino, Parton and Salamon in [53, Theorem 1.2]. If we denote by *FPS-nilmanifold* any such a manifold, we prove the following result.

Theorem (see Theorem 6.4.2)**.** *Let (M, J) be a FPS-nilmanifold. Then, any left-invariant (SKT) metric is geometrically-Bott-Chern-formal.*

Moreover, we extend this result to a class of nilmanifolds which are a generalization of FPS-manifolds in a arbitrary higher dimension (see Theorem 6.4.4).

Contrarily to the mentioned positive results, on a compact complex manifold the existence of a SKT metric does not imply the existence of geometrically-Bott-Chern-formal metrics. More precisely, we prove this for the product of a pair of certain compact complex surfaces by providing a non vanishing Aeppli-Bott-Chern-Massey product on each manifold.

Theorem (see Theorem 6.4.5)**.** *Let (M, J) be the product of either two Kodaira surfaces, two Inoue surfaces, or a Kodaira surface and a Inoue surface. Then (M, J) admits SKT metrics but does not admit geometrically-Bott-Chern-formal metrics.*

Furthermore, a similar result holds also for manifolds which are not a product of manifolds, as it is shown for a family of nilmanifolds of complex dimension 4 in Theorem 6.4.6.

6.1 p-pluriclosed forms

In order to recall the characterization theorem of compact complex manifolds admitting a p-pluriclosed structure (see Section 1.5), we review some known facts on positive currents.

Let M be an n-dimensional complex manifold and let $A^{p,q}(\Omega)$ respectively $\mathcal{D}^{p,q}(\Omega)$) be the space of (p,q)-forms respectively (p,q)-forms with compact support on M. Consider the $\mathcal{C}^\infty$-topology on $\mathcal{D}^{p,q}(M)$. The *space of currents* of *bi-dimension* (p,q) or of *bi-degree* $(n-p,n-q)$ is the topological dual $\mathcal{D}'_{p,q}(M)$ of $\mathcal{D}^{p,q}(M)$. A current of bi-dimension (p,q) on M can be identified with a $(n-p,n-q)$-form on M with coefficients distributions. A current T of bi-dimension (p,p) is said to be *real* if $T(\eta)=T(\overline{\eta})$, for any $\eta \in \mathcal{D}^{p,q}(M)$. A real current $T \in \mathcal{D}'_{p,p}(M)$ is said to be *strongly positive* if,

$$T(\Omega) \geq 0,$$

for every weakly positive (p,p)-form Ω. We have the following (see [4, Theorem 2.4,(4)])

Theorem 6.1.1. *A compact n-dimensional complex manifold N has a strictly weakly positive (p,p)-form Ω with $\partial\overline{\partial}\Omega=0$ if and only if N has no strongly positive currents $T \neq 0$ of bidimension (p,p), such that $T = i\partial\overline{\partial}A$ for some current A of bidimension $(p+1,p+1)$.*

The following simple yet useful lemma yields an obstruction to the existence of p-pluriclosed forms on a closed almost complex manifold.

Lemma 6.1.2 ([133]). *Let (M,J) be a closed almost complex manifold of real dimension $2n$. Let α be a $(2n-2p-2)$-form which is not dd^c-closed and such that*

$$(dd^c\alpha)^{n-p,n-p} = \sum c_k \psi^k \wedge \overline{\psi}^k,$$

with ψ^k simple $(n-p,0)$-covectors and $c_k \neq 0$ constants having the same sign. Then (M,J) does not admit a p−pluriclosed form.

In particular,

- *for $p=1$, (M,J) does not admit SKT metrics;*
- *for $p=n-2$, (M,J) does not admit astheno-Kähler metrics.*

Proof. We prove this lemma by contradiction. Suppose there exists a p-pluriclosed form Ω on (M,J), i.e., Ω is a (p,p)-real form which is dd^c-closed and, for every $x \in M$, $\Omega_x \in \bigwedge^{p,p}(T_xM^*)$ is transverse. Then, let α be a $(2n-2p-2)$-form on (M,J) as above and let us assume, for example, that each $c_k > 0$. Since M is closed, by Stokes theorem we have that

$$0 = \int_M d(d^c(\sigma_n\Omega\wedge\alpha)) = \int_M \sigma_n\Omega\wedge dd^c\alpha = \sum_k c_k \int_M \sigma_n\Omega\wedge\psi^k\wedge\overline{\psi}^k > 0,$$

which is a contradiction. To end the proof, notice that if F is an astheno-Kähler metric on (M,J), the $(n-2,n-2)$-form F^{n-2} is a $(n-2)$-pluriclosed form on (M,J). Analogously, if F is a SKT metric on (M,J), F is 1-pluriclosed form on (M,J). □

Remark 6.1.3. In Lemma 6.1.2 the thesis on the non existence of Hermitian metrics satisfying $dd^cF=0$, for $p=1$, respectively $dd^cF^{n-2}=0$, for $p=n-2$, is still valid, without assuming the integrability of J.

6.2 Astheno-Kähler metrics on 5-dimensional nilmanifolds

We now proceed to construct a family of nilmanifolds of complex dimension 5 endowed with a left-invariant complex structure admitting an astheno-Kähler metric.

Let $\{\eta^1, \dots, \eta^5\}$ be the set of complex forms of type $(1,0)$, such that

$$\begin{cases} d\eta^j &= 0,\ j = 1, \dots, 4, \\ d\eta^5 &= a_1\,\eta^{12} + a_2\,\eta^{13} + a_3\,\eta^{13} + a_4\,\eta^{1\bar{1}} + a_5\,\eta^{1\bar{2}} + a_6\,\eta^{1\bar{3}} + a_7\,\eta^{1\bar{4}} \\ & +b_1\,\eta^{23} + b_2\,\eta^{24} + b_3\,\eta^{2\bar{1}} + b_4\,\eta^{2\bar{2}} + b_5\,\eta^{2\bar{3}} + b_6\,\eta^{2\bar{4}} \\ & +c_1\,\eta^{34} + c_2\,\eta^{3\bar{1}} + c_3\,\eta^{3\bar{2}} + c_4\,\eta^{3\bar{3}} + c_5\,\eta^{3\bar{4}} \\ & +d_1\,\eta^{4\bar{1}} + d_2\,\eta^{4\bar{2}} + d_3\,\eta^{4\bar{3}} + d_4\,\eta^{4\bar{4}} \end{cases} \tag{6.2.1}$$

where $a_h, b_k, c_r, d_s \in \mathbb{C}$, $h = 1, \dots, 7$, $k = 1, \dots, 6$, $r = 1, \dots, 5$, $s = 1, \dots, 4$ and we set as usual $\eta^{AB} = \eta^A \wedge \eta^B$. Then, setting $\mathfrak{g}^{1,0} = \mathrm{Span}\langle \eta^1, \dots, \eta^5 \rangle$, we obtain that $\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}}$ gives rise to an integrable almost complex structure J on the real 2-step nilpotent Lie algebra $\mathfrak{g}$. Let G be the simply-connected and connected Lie group with Lie algebra $\mathfrak{g}$. Then, for any given choice of parameters $a_h, b_k, c_r, d_s \in \mathbb{Q}[i]$ as a consequence of Malcev's theorem [98], there exist lattices $\Gamma \subset G$, so that $(M = \Gamma \backslash G, J)$ is a nilmanifold endowed with a complex structure J with $\dim_{\mathbb{C}} M = 5$. We have the following

Theorem 6.2.1 ([133]). *Let $M = \Gamma \backslash G$ and J be the complex structure on M defined by (6.2.1). Then*

I) *The diagonal metric g on (M, J) whose fundamental form is*

$$F = \frac{i}{2} \sum_{h=1}^{5} \eta^{h\bar{h}}$$

is astheno-Kähler if and only if the following condition holds

$$\begin{aligned} 2\mathfrak{Re}\,(d_4\bar{a}_4 + d_4\bar{b}_4 + d_4\bar{c}_4 + c_4\bar{a}_4 + c_4\bar{b}_4 + b_4\bar{a}_4) &= |a_1|^2 + |a_2|^2 + |a_3|^2 + |a_5|^2 + |a_6|^2 + |a_7|^2 + \\ &+ |b_1|^2 + |b_2|^2 + |b_3|^2 + |b_5|^2 + |b_6|^2 + \\ &+ |c_1|^2 + |c_2|^2 + |c_3|^2 + |d_1|^2 + |d_2|^2. \end{aligned} \tag{6.2.2}$$

II) *Let*

$$a_2 = a_3 = a_5 = a_6 = a_7 = b_1 = b_2 = b_3 = b_5 = b_6 = c_2 = c_3 = c_5 = d_1 = d_2 = d_3 = 0.$$

Then the metric g satisfies $dd^cF^3 = 0$ and $dd^cF^2 = 0$ if and only if the following conditions hold

$$\begin{cases} 2\,\mathfrak{Re}\,(d_4\bar{a}_4 + d_4\bar{b}_4 + d_4\bar{c}_4 + c_4\bar{a}_4 + c_4\bar{b}_4 + b_4\bar{a}_4) = |a_1|^2 + |c_1|^2 \\ 2\,\mathfrak{Re}\,(c_4\bar{a}_4 + c_4\bar{b}_4 + b_4\bar{a}_4) = |a_1|^2 \\ \mathfrak{Re}\,(c_4\bar{b}_4 - d_4\bar{a}_4) = 0 \\ \mathfrak{Re}\,(b_4\bar{d}_4 - c_4\bar{a}_4) = 0. \end{cases} \tag{6.2.3}$$

Proof. As for I), with the aid of Sagemath and structure equations (6.2.1), it is easy to the see that

$$\begin{aligned}\tfrac{2}{3}dd^cF^3 \;=\; & \big(2\mathfrak{Re}\,(d_4\bar a_4 + d_4\bar b_4 + d_4\bar c_4 + c_4\bar a_4 + c_4\bar b_4 + b_4\bar a_4) \\ & -|a_1|^2 - |a_2|^2 - |a_3|^2 - |a_5|^2 - |a_6|^2 - |a_7|^2 \\ & -|b_1|^2 - |b_2|^2 - |b_3|^2 - |b_5|^2 - |b_6|^2 \\ & -|c_1|^2 - |c_2|^2 - |c_3|^2 - |d_1|^2 - |d_2|^2\,\big)\,\eta^{1234\bar1\bar2\bar3\bar4},\end{aligned} \tag{6.2.4}$$

i.e., the metric F is astheno-Kähler on (M, J) if and only if (6.2.2) holds.
II) Under the assumption

$$a_2 = a_3 = a_5 = a_6 = a_7 = b_1 = b_2 = b_3 = b_5 = b_6 = c_1 = c_2 = c_3 = c_5 = d_1 = d_2 = d_3 = 0,$$

taking into account (6.2.4) and by a straightforward computation, we obtain that

$$dd^cF^3 = 0, \quad dd^cF^2 = 0$$

if and only if

$$\begin{cases} 2\mathfrak{Re}\,(d_4\bar a_4 + d_4\bar b_4 + d_4\bar c_4 + c_4\bar a_4 + c_4\bar b_4 + b_4\bar a_4) - |a_1|^2 - |c_1|^2 = 0 \\ 2\mathfrak{Re}\,(c_4\bar a_4 + c_4\bar b_4 + b_4\bar a_4) - |a_1|^2 = 0 \\ 2\mathfrak{Re}\,(d_4\bar a_4 + d_4\bar b_4 + b_4\bar a_4) - |a_1|^2 = 0 \\ 2\mathfrak{Re}\,(d_4\bar a_4 + d_4\bar c_4 + c_4\bar a_4) - |c_1|^2 = 0 \\ 2\mathfrak{Re}\,(d_4\bar b_4 + d_4\bar c_4 + c_4\bar b_4) - |c_1|^2 = 0. \end{cases}$$

The last system is equivalent to (6.2.3). □

Remark 6.2.2. Recall that an Hermitian metric g on a n-dimensional complex manifold (M, J) is said to be *balanced* if its fundamental form ω satisfies $d\omega^{n-1} = 0$. In [146, p. 185] the authors asked for an example of a non-Kähler compact complex manifold which admits both balanced and astheno-Kähler metrics. In [52], and independently in [92], the authors constructed explicit examples of such manifolds in any dimension. As a direct application of Theorem 6.2.1, we obtain families of 5-dimensional complex nilmanifolds carrying both astheno-Kähler and balanced metrics. We apply a similar construction as in [92, Remark 2.6]. Let

$$\hat F = \frac{i}{2}(A\eta^{1\bar1} + \eta^{2\bar2} + \eta^{3\bar3} + \eta^{4\bar4} + \eta^{5\bar5})$$

where A is a positive real number. Then $d\hat F^4 = 0$ if and only if

$$a_4 + Ab_4 + Ac_4 + Ad_4 = 0, \tag{6.2.5}$$

where a_4, b_4, c_4, d_4 are the parameters as in (6.2.1) Let g be the diagonal metric whose fundamental form is

$$F = \frac{i}{2}(\eta^{1\bar1} + \eta^{2\bar2} + \eta^{3\bar3} + \eta^{4\bar4} + \eta^{5\bar5}).$$

Then, according to I) of Theorem 6.2.1, g is astheno-Kähler if and only if condition (6.2.2) holds. Take

$$a_4 = -\frac{1}{10}(1 + 2i), \qquad b_4 = i, \qquad c_4 = i, \qquad d_4 = 1, \qquad A = \frac{1}{10}.$$

Then, with this choice of parameters, we obtain

$$a_4 + Ab_4 + Ac_4 + Ad_4 = -\frac{1}{10} - \frac{1}{5}i + \frac{1}{10}i + \frac{1}{10}i + \frac{1}{10} = 0,$$

that is (6.2.5) is satified and so, for such a choice of parameters, $\hat{F}$ gives rise to a balanced metric on $M = \Gamma\backslash G$. A straightforward calculation yields

$$2\mathfrak{Re}\,(d_4\bar{a}_4 + d_4\bar{b}_4 + d_4\bar{c}_4 + c_4\bar{a}_4 + c_4\bar{b}_4 + b_4\bar{a}_4) = 1.$$

Therefore, the Hermitian metric g is astheno-Kähler if and only if condition (6.2.2) reads as

$$\begin{aligned} 1 &= |a_1|^2 + |a_2|^2 + |a_3|^2 + |a_5|^2 + |a_6|^2 + |a_7|^2 + \\ &+ |b_1|^2 + |b_2|^2 + |b_3|^2 + |b_5|^2 + |b_6|^2 + \\ &+ |c_1|^2 + |c_2|^2 + |c_3|^2 + |d_1|^2 + |d_2|^2 . \end{aligned} \tag{6.2.6}$$

One can check that there exist solutions in $\mathbb{Q}[i]$ of equation (6.2.6), so that, for any given solution, the associated complex nilmanifold defined as in (6.2.1) admits both a balanced metric and an astheno-Kähler metric.

As an application of Lemma 6.1.2, we provide a family of compact almost complex nilmanifolds without 2-pluriclosed forms.

Proposition 6.2.3 ([133])**.** *Let $\{\psi^1, \dots, \psi^4\}$ be the set of complex forms of type $(1,0)$, such that*

$$\begin{cases} d\psi^j &= 0, \quad j = 1, \dots, 3, \\ d\psi^4 &= a_1\,\psi^{12} + a_2\,\psi^{23} + a_3\,\psi^{1\bar{1}} + a_4\,\psi^{2\bar{2}} + a_5\,\psi^{3\bar{3}} + a_6\,\psi^{\bar{1}\bar{2}} + a_7\,\psi^{\bar{2}\bar{3}}, \end{cases} \tag{6.2.7}$$

where $a_1, \dots, a_7 \in \mathbb{Q}[i]$. Let G be the corresponding simply-connected and connected nilpotent Lie group and $\Gamma \subset G$ be a lattice such that $N = \Gamma\backslash G$ is a compact nilmanifold. Assume that

$$a_1\overline{a_2} + \overline{a_6}a_7 = 0 \tag{6.2.8}$$

and set $a = (a_1, \dots, a_7)$. Then (N, J_a) does not admit any 2-pluriclosed form.

Proof. A straightforward calculation using (6.2.7) yields to

$$\begin{aligned} \tfrac{i}{2}dd^c\psi^{4\bar{4}} &= (|a_1|^2 + |a_6|^2)\psi^{12\bar{1}\bar{2}} + (|a_2|^2 + |a_7|^2)\psi^{23\bar{2}\bar{3}} + (a_1\overline{a_2} + \overline{a_6}a_7)\psi^{12\bar{2}\bar{3}} \\ &\quad + (a_2\overline{a_1} + \overline{a_7}a_6)\psi^{23\bar{1}\bar{2}} \\ &= (|a_1|^2 + |a_6|^2)\psi^{12\bar{1}\bar{2}} + (|a_2|^2 + |a_7|^2)\psi^{23\bar{2}\bar{3}} \end{aligned}$$

The thesis follows immediately from Lemma 6.1.2. ∎

Remark 6.2.4. For any given a such that $(a_6, a_7) \neq (0,0)$, J_a is a non integrable almost complex structure on N. Consequently, for such an a, (N, J_a) is an almost complex manifold with no 2-pluriclosed forms.

6.3 Blow-ups of astheno-Kähler metrics

By classical results and more recent ones, (see [27, 154, 4, 54]), we know that, for compact complex manifolds, the property of admitting, respectively, Kähler, balanced, or SKT metrics, is stable under blow-ups either in a point or along a compact complex submanifold. Regarding astheno-Kähler metrics, in [55], it is proved the following result.

Proposition 6.3.1. *([55, Proposition 2.4]) Let (M, J, g) be an astheno-Kähler manifold of complex dimension n such that its fundamental 2-form F satisfies*

$$dd^cF = 0, \quad dd^cF^2 = 0. \tag{6.3.1}$$

Then both the blow-up $\tilde{M}_p$ of M at a point $p \in M$ and the blow-up $\tilde{M}_Y$ of M along a compact complex submanifold Y admit an astheno-Kähler metric satisfying (6.3.1), too.

In this section, we will show that blow-ups of astheno-Kähler metrics do not preserve additional differential properties of the metric, namely we construct an example of a 5-dimensional manifold M admitting a metric F satisfying

$$dd^cF^2 = 0, \quad dd^cF^3 = 0, \tag{6.3.2}$$

and we will consider the blow-up of such manifold along a submanifold. We will prove that such blow-up does not admit any Hermitian metric $\tilde{F}$ which satisfies $dd^c\tilde{F}^2 = 0$ and $dd^c\tilde{F}^3 = 0$.

We note that if $dd^cF = 0$, conditions (6.3.1) of [55] would be verified, thus yielding stability. Therefore, when we consider a Hermitian metric F which satisfies weaker conditions than (6.3.1), e.g., the astheno-Kähler condition and the differential condition $dd^cF^{n-3} = 0$, in general such conditions are not stable under blow-ups

Now, we construct a family of 5-dimensional compact complex nilmanifolds endowed with a Hermitian metric whose fundamental form F satisfies (6.3.2) and such that the blow-up of M along a suitable 3-dimensional complex submanifold Y has no Hermitian metrics satisfying (6.3.2). To this purpose, we start by considering the following nilpotent Lie group $G := (\mathbb{C}^5, *)$, where the operation $*$ is defined for every $w = (w_1, w_2, w_3, w_4, w_5), z = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5$ by

$$\begin{aligned}&w * z := \\ &(w_1 + z_1, w_2 + z_2, w_3 + z_3, w_4 + z_4, z_5 + a_1w_1z_2 + a_4\overline{w}_1z_1 + b_4\overline{w}_2z_2 + c_1w_3z_4 + c_4\overline{w}_3z_3 + d_4\overline{w}_4z_4 + w_5),\end{aligned}$$

with $a_1, a_4, b_4, c_1, c_4, d_4 \in \mathbb{Q}[i]$. We can then consider the following forms on G

$$\begin{cases}\eta^i = dz_i, \quad i \in \{1,2,3,4\} \\ \eta^5 = dz_5 - a_1z_1dz_2 - a_4\overline{z}_1dz_1 - b_4\overline{z}_2dz_2 - c_1z_3dz_4 - c_4\overline{z}_3dz_3 - d_4\overline{z}_4dz_4.\end{cases}$$

It can be easily seen that $\{\eta^1, \dots, \eta^5\}$ are left-invariant global forms on G with structure equations

$$\begin{cases}d\eta^i = 0, \quad i \in \{1,2,3,4\} \\ d\eta^5 = -a_1\eta^{12} + a_4\eta^{1\overline{1}} + b_4\eta^{2\overline{2}} - c_1\eta^{34} + c_4\eta^{3\overline{3}} + d_4\eta^{4\overline{4}}.\end{cases}$$

The dual left-invariant complex vectors fields $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$ on G are given by

$$\begin{cases} Z_1 = \frac{\partial}{\partial z_1} + a_4 \overline{z}_1 \frac{\partial}{\partial z_5} \\ Z_2 = \frac{\partial}{\partial z_2} + (a_1 z_1 + b_4 \overline{z}_2) \frac{\partial}{\partial z_5} \\ Z_3 = \frac{\partial}{\partial z_3} + c_4 \overline{z}_3 \frac{\partial}{\partial z_5} \\ Z_4 = \frac{\partial}{\partial z_4} + (c_1 z_3 + d_4 \overline{z}_4) \frac{\partial}{\partial z_5} \\ Z_5 = \frac{\partial}{\partial z_5}. \end{cases}$$

We note that $T_{\mathbb{C}}G = \langle Z_1, \dots, Z_5, \overline{Z}_1, \dots, \overline{Z}_5 \rangle$ and the distribution $D = \langle Z_1, \dots, Z_5 \rangle$ is integrable. Therefore, if we denote by J the almost complex structure on G for which $\{Z_1, \dots, Z_5\}$ is a frame of $(1,0)$-vector fields and $\{\eta^1, \dots, \eta^5\}$ is a coframe of $(1,0)$-forms, then J is an integrable left-invariant almost complex structure on G.

Since the constant structures $a_1, a_4, b_4, c_1, c_4, d_4$ are numbers in $\mathbb{Q}[i]$, Malcev theorem assures the existence of a discrete uniform subgroup Γ such that $M := \Gamma \backslash G$ is a compact nilmanifold. In particular, since J is left-invariant on G, it descends to M, i.e., (M, J) is a complex 5-dimensional nilmanifold. In particular $\{Z_1, \dots, Z_5\}$ and $\{\eta^1, \dots, \eta^5\}$ are a global left-invariant frame of $(1,0)$-vector fields, respectively $(1,0)$-forms on M.

In particular, we point out that M is the nilmanifold associated to the Lie algebra $\mathfrak{g}$ of Section 4, with structure constants

$$a_2 = a_3 = a_5 = a_6 = a_7 = b_1 = b_2 = b_3 = b_5 = b_6 = c_2 = c_3 = c_5 = d_1 = d_2 = d_3 = 0.$$

If we denote by

$$p \colon G \to M$$

the natural quotient projection from G to $\Gamma \backslash G$ and we set

$$Y_0 := \{(z_1, z_2, z_3, z_4, z_5) : z_2 = z_4 = 0\} \subset G,$$

then $p(Y_0) =: Y \subset M$ is a compact complex 3-dimensional submanifold of M whose complexified tangent bundle $T_{\mathbb{C}}Y$ is spanned by $\{Z_1, Z_3, Z_5, \overline{Z}_1, \overline{Z}_3, \overline{Z}_5\}$.

It is immediate to check that Y is a 3-dimensional nilmanifold and $\{\eta^1, \eta^3, \eta^5\}$ is a global coframe of $(1,0)$-forms on Y with complex structure equations given by

$$\begin{cases} d\eta^1 = 0, \\ d\eta^3 = 0 \\ d\eta^5 = a_4 \eta^{1\overline{1}} + c_4 \eta^{3\overline{3}}. \end{cases} \tag{6.3.3}$$

For the convenience of the reader, we set $\alpha^1 := \eta^1$, $\alpha^2 := \eta^3$, and $\alpha^3 := \eta^5$, so that we can rewrite (6.3.3) as

$$\begin{cases} d\alpha^1 = 0, \\ d\alpha^2 = 0 \\ d\alpha^3 = a_4 \alpha^{1\overline{1}} + c_4 \alpha^{2\overline{2}}. \end{cases} \tag{6.3.4}$$

Now fix the following constant structures

$$a_1 = -1 - 3i, \quad a_4 = 1, \quad b_4 = 1, \quad c_1 = -4, \quad c_4 = 2, \quad d_4 = 2,$$

and consider the metric

$$F = \frac{i}{2}\sum_{j=1}^{5}\eta^j \wedge \eta^{\bar{j}}.$$

For such choice of coefficients, by Theorem 6.2.1, we have that

$$dd^cF^2 = 0, \quad dd^cF^3 = 0, \quad dd^cF \neq 0.$$

Now, let us consider the blow-up $\pi\colon \tilde{M}_Y \to M$ of M along the compact complex submanifold Y, with E the exceptional divisor. We note that E has complex dimension 4, since each fiber $\pi^{-1}(y) \subset \tilde{M}_Y$ over a point $y \in Y$ has dimension 1 and $\dim_{\mathbb{C}} Y = 3$.

By contradiction, now let us assume that the astheno-Kähler condition $dd^cF^3 = 0$ and the condition $dd^cF^2 = 0$ are stable, i.e., there exists a Hermitian metric on $\tilde{M}_Y$ such that $dd^c\tilde{F}^3 = 0$ and $dd^c\tilde{F}^2 = 0$.

Then, the restriction of $\tilde{F}$ on E gives rise to a astheno-Kähler metric on E, that is $dd^c(\tilde{F}|_E)^2 = 0$, i.e., E is a 2-pluriclosed manifold.

We now recall the following useful proposition by Alessandrini ([3, Proposition 3.1]), adapted here to the setting of p-pluriclosed manifolds.

Proposition 6.3.2. *Let M and N be connected compact complex manifolds, with $\dim N = n > m = \dim M \geq 1$, and let $f\colon N \to M$ be a holomorphic submersion, where $a := n - m = \dim f^{-1}(x), x \in M$, is the dimension of the standard fibre F. If N is p-pluriclosed for some p, $a < p \leq n-1$, then M is $(p-a)$-pluriclosed.*

Let us consider the map $\pi|_E\colon E \to Y$. We note that $\pi|_E$ is a holomorphic submersion with 1-dimensional fibers, therefore by Proposition 6.3.2, we have that Y is 1-pluriclosed, i.e., it admits a SKT metric.

However, this is absurd by either the characterization of 3-dimensional SKT nilmanifolds by [53], or Lemma 6.1.2, observing that $dd^c(-\alpha^{3\bar{3}}) = 8\alpha^{12\bar{1}\bar{2}}$.

Summing up, we have proved the following

Theorem 6.3.3 ([133])**.** *On a compact complex manifold of dimension n, the existence of a Hermitian metric F such that*

$$dd^cF^{n-2} = 0, \qquad dd^cF^{n-3} = 0$$

is not preserved by blow-up.

6.4 Geometric Bott-Chern formality and Strong Kähler with Torsion metrics

In this section we investigate the relation between the notions of SKT metrics and geometrically-Bott-Chern-formal metrics in the setting of nilmanifolds endowed with a left-invariant complex structure J and a Hermitian metric g.

In complex dimension 3, the existence of SKT metrics is fully characterized by Fino, Parton, and Salamon, in terms of the complex structure equation of the manifold, as we recall in the following.

Theorem 6.4.1. *([53, Theorem 1.2]). Let $M = \Gamma\backslash G$ be a 6-dimensional nilmanifold with an invariant complex structure J. Then the SKT condition is satisfied by either all invariant Hermitian metrics g or by none. Indeed, it is satisfied if and only if J has a basis (α^i) of $(1,0)$-forms such that*

$$\begin{cases} d\alpha^1 = 0 \\ d\alpha^2 = 0 \\ d\alpha^3 = A\alpha^{\bar{1}2} + B\alpha^{\bar{2}2} + C\alpha^{1\bar{1}} + D\alpha^{1\bar{2}} + E\alpha^{12} \end{cases} \tag{6.4.1}$$

where A, B, C, D, E are complex numbers such that

$$|A|^2 + |D|^2 + |E|^2 + 2\mathfrak{Re}\left(\overline{B}C\right) = 0. \tag{6.4.2}$$

We will refer to 6-dimensional nilmanifolds satisfying (6.4.1) and (6.4.2) as *Fino-Parton-Salamon-nilmanifolds*, shortly *FPS-nilmanifolds* and we will denote the Lie algebra of the group G by the symbol $\mathfrak{g}$.

By this classification result, we are able to prove the following theorem.

Theorem 6.4.2 ([133])**.** *Let (M, J) be an FPS-nilmanifold. Then, any left-invariant (SKT) metric is geometrically-Bott-Chern-formal.*

Before proving Theorem 6.4.2, we will need the following lemma for the $\partial\overline{\partial}$ operator on this class of manifolds.

Lemma 6.4.3. *Let (M, J) be a FPS-nilmanifold. Then,*

$$\partial\overline{\partial}|_{\wedge^{p,q}\mathfrak{g}} \equiv 0.$$

Proof. (of Lemma 6.4.3). We begin by observing that it suffices to prove that $\partial\overline{\partial}\alpha^{3\overline{3}} = 0$. In fact, let us consider the left-invariant (p, q)-form on M

$$\sigma := \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_p} \wedge \alpha^{\overline{j}_1} \wedge \ldots \alpha^{\overline{j}_q}.$$

We note that if σ does not contain $\alpha^{3\overline{3}}$, them $\partial\overline{\partial}\sigma = 0$. In fact, let us consider the two cases:
(1) $i_k \neq 3, \overline{j}_l \neq \overline{3}$ for every $k \in \{1, \ldots, p\}$, $l \in \{1, \ldots, q\}$.
(2) $i_k = 3$ for some $k \in \{1, \ldots, p\}$ and $\overline{j}_l \neq \overline{3}$ for every $l \in \{1, \ldots, q\}$, or $i_k \neq 3$ for every $k \in \{1, \ldots, p\}$ and $\overline{j}_l = \overline{3}$ for some $l \in \{1, \ldots, q\}$.

In case (1), by structure equations (6.4.1) we immediately have that $\overline{\partial}\alpha^{i_k} = \overline{\partial}\alpha^{\overline{j}_l} = 0$. Hence, by Leibniz rule, $\partial\overline{\partial}\sigma = \partial(\overline{\partial}\sigma) = 0$.

For case (2), let $i_k = 3$ for $k \in \{1, \ldots, p\}$ and $\overline{j}_l \neq \overline{3}$. Then, up to a sign change, by Leibniz rule we have that $\overline{\partial}\sigma = \overline{\partial}\alpha^3 \wedge \hat{\sigma}$, where $\hat{\sigma}$ is σ from which we remove α^3. Since $\overline{\partial}\alpha^3 = A\alpha^{\overline{1}2} + B\alpha^{2\overline{2}} + C\alpha^{1\overline{1}} + D\alpha^{1\overline{2}}$, we can write that

$$\overline{\partial}\sigma = A\alpha^{\overline{1}2} \wedge \hat{\sigma} + B\alpha^{2\overline{2}} \wedge \hat{\sigma} + C\alpha^{1\overline{1}} \wedge \hat{\sigma} + D\alpha^{1\overline{2}} \wedge \hat{\sigma}.$$

Since $\overline{\partial}\sigma$ does not contain α^3 or $\alpha^{\overline{3}}$, once again by (6.4.1) and Leibniz rule, we obtain $\partial\overline{\partial}\sigma = \partial(\overline{\partial}\sigma) = 0$. Analogous computations can be carried out when $i_k \neq 3$ for every $k \in \{1, \ldots, p\}$ and $\overline{j}_l = \overline{3}$ for some $l \in \{1, \ldots, q\}$.

Let us then consider $\partial\overline{\partial}\alpha^{3\overline{3}}$. If g is any left-invariant metric on (M, J) with fundamental associated form

$$F = \frac{i}{2}\sum_{k=1}^{3} F_{k\overline{k}}\alpha^{k\overline{k}} + \frac{1}{2}\sum_{k<h}\left(F_{k\overline{h}}\alpha^{k\overline{h}} - \overline{F}_{k\overline{h}}\alpha^{h\overline{k}}\right)$$

then, by the above argument $\partial\overline{\partial}F = \frac{i}{2}F_{3\overline{3}}\partial\overline{\partial}\alpha^{3\overline{3}}$. By Theorem 6.4.1, any left-invariant Hermitian metric on (M, J) is SKT, therefore, g is SKT, i.e., $\partial\overline{\partial}F = \partial\overline{\partial}\alpha^{3\overline{3}} = 0$.

Therefore, up to swapping the forms and changing the sign accordingly, for a left-invariant form $\sigma = \alpha^{3\overline{3}} \wedge \hat{\sigma}$ with $\hat{\sigma}$ not containing α^3 nor $\alpha^{\overline{3}}$, for the above arguments we have that $\partial\overline{\partial}\sigma = \partial\overline{\partial}(\alpha^{3\overline{3}}) \wedge \hat{\sigma} = 0$. Then, by linearity of the $\partial\overline{\partial}$ operator, we can conclude. $\square$

Proof (of Theorem 6.4.2). First of all, we observe that the complex structure J on the nilmanifold $M := \Gamma\backslash G$ is nilpotent, i.e., there exists a basis of $(1,0)$-forms $\{\alpha_i\}_{i=1}^3$, such that $d\alpha \in \mathrm{Span}_{\mathbb{C}}\langle\alpha^{ij}, \alpha^{i\bar{j}}\rangle_{i,j=1}^2$. Hence, [11, Theorem 3.7] (see also [124, Corollary 3.12]) yields the isomorphisms

$$H^{p,q}_{BC}(\mathfrak{g}, J) \hookrightarrow H^{p,q}_{BC}(M), \tag{6.4.3}$$

i.e., the Bott-Chern cohomology of (M, J) can be computed via the subcomplex of left-invariant forms.

Now, let g be a left-invariant metric on (M, J) with fundamental associated form F. We will show that g is geometrically-Bott-Chern-formal. Let us then fix two Bott-Chern harmonic forms $\beta \in \mathcal{H}^{p,q}_{BC}(M, g)$, $\gamma \in \mathcal{H}^{r,s}_{BC}(M, g)$. Then, the product $\beta \wedge \gamma$ is Bott-Chern harmonic with respect to g if, and only if,

$$d(\beta \wedge \gamma) = 0, \quad \partial\overline{\partial} *_g (\beta \wedge \gamma) = 0.$$

By Leibnitz rule, $d(\beta \wedge \gamma) = 0$ since both β and γ are Bott-Chern harmonic. Moreover, by Lemma 6.4.3, $\partial\overline{\partial}(*_g\beta \wedge \gamma) = 0$, i.e., $\beta \wedge \gamma \in \mathcal{H}^{p+r,q+s}_{BC}(M, g)$. Hence, g is a geometrically-Bott-Chern -formal metric on (M, J). □

A similar result also holds for a class of manifolds which generalizes the FPS manifolds in higher dimensions.

Theorem 6.4.4 ([133]). *Let M be any $2n$-dimensional nilmanifold endowed with an invariant integrable almost complex structure J induced by a coframe $\{\eta^1, \dots, \eta^n\}$ of left-invariant $(1,0)$-forms on (M, J) with structure equations given by*

$$\begin{cases} d\eta^i = 0, \quad i \in \{1, \dots, n-1\}, \\ d\eta^n \in Span\langle\eta^{ij}, \eta^{i\bar{j}}\rangle_{i,j=1,\dots,n-1}. \end{cases}$$

Then, any invariant SKT metric is geometrically-Bott-Chern-formal.

Proof. In a similar fashion to proof of Theorem 6.4.2, it can be shown that, if there exists a left-invariant SKT metric on (M, J), then $\partial\overline{\partial}\eta^{n\bar{n}} = 0$, and in particular the $\partial\overline{\partial}$ operator vanishes on any left-invariant form on (M, J). Notice that the Bott-Chern cohomology of (M, J) can be computed via the subcomplex of left-invariant forms and its Bott-Chern harmonic representatives are invariant since the complex structure J is nilpotent and [11, Theorem 3.8] applies. Hence, if g is a SKT metric on (M, J) and we take two Bott-Chern harmonic forms α and β of bedegree (p, q), respectively (r, s), then α and β are left-invariant and $\alpha \wedge \beta \in \bigwedge^{p+r,q+s}\mathfrak{g}$ satisfies $d(\alpha \wedge \beta) = 0$ and by, structure equations, $\partial\overline{\partial}(*_g\alpha \wedge \beta) = 0$. Therefore $\alpha \wedge \beta$ is Bott-Chern harmonic, i.e., the product of two left-invariant Bott-Chern harmonic forms with respect to g is Bott-Chern harmonic. This implies that every left-invariant SKT metric is geometrically-Bott-Chern-formal.

□

In higher dimension and under more general conditions on the complex structure of the nilmanifold, however, similar results do not hold. Certain products of compact complex surfaces, e.g., are SKT but do not admit geometrically-Bott-Chern-formal metrics, as proved in the following theorem.

Theorem 6.4.5 ([133]). *Let (M, J) be the product of either two Kodaira surfaces, two Inoue surfaces, or a Kodaira surface and a Inoue surface. Then (M, J) admits SKT metrics but does not admit geometrically-Bott-Chern-formal metrics.*

Proof. We begin by noticing that given the product of any two of the above compact complex surfaces $(M,J)=(M',J')\times(M'',J'')$, such manifold admits an SKT metric.

Let us consider the product metric $g := g'+g''$, given by the sum of the diagonal constant metrics g' and g'' with respect to certain coframes $\{\eta^1,\eta^2\}$ and $\{\eta^3,\eta^4\}$ on, respectively, (M',J') and (M'',J'') and let

$$F'=\frac{i}{2}(\eta^{1\bar{1}}+\eta^{2\bar{2}}),\quad F''=\frac{i}{2}(\eta^{3\bar{3}}+\eta^{4\bar{4}})$$

be the fundamental forms associated to, respectively, g' and g''. By a dimension argument, we have that on each factor

$$\partial\bar{\partial}F'=0,\quad \partial\bar{\partial}F''=0.$$

Therefore, if $F := F'+F''$, it is clear that

$$\partial\bar{\partial}F=\partial\bar{\partial}F'+\partial\bar{\partial}F''=0,$$

i.e., the product metric g is SKT on (M,J). (We will refer to such metric by g.)

We will show that none of the above product manifolds admits geometrically-Bott-Chern-formal metrics by exhibiting a non vanishing Aeppli-Bott-Chern-Massey product on each manifold.

Note that on each product, Bott-Chern and Aeppli cohomologies can be computed via the subcomplex of invariant complex forms, as follows. First of all, the de Rham cohomology and the Dolbeault cohomology of compact surfaces diffeomorphic to solvmanifolds can be computed in terms of invariant forms, see, e.g., [12]. Therefore, applying Kunneth formula, it follows that the de Rham and the Dolbeault cohomologies of the product of any two such surfaces can be computed in terms of the invariant forms. By [11, Theorem 3.7], also the Bott-Chern and Aeppli cohomologies can be computed in terms of the invariant forms.

(i) **The product of two Kodaira surfaces of primary type.**
Let $(M,J)=(KT,J_{KT})\times(KT,J_{KT})$ be the product of two Kodaira surfaces. The complex structure J is determined by the coframe $\{\eta^1,\eta^2,\eta^3,\eta^4\}$ of left-invariant $(1,0)$-forms such that its structure equations read

$$\begin{cases} d\eta^1=0\\ d\eta^2=A\eta^{1\bar{1}}\\ d\eta^3=0\\ d\eta^4=B\eta^{3\bar{3}}, \end{cases} \tag{6.4.4}$$

for $A,B\in\mathbb{C}\smallsetminus\{0\}$.

From (6.4.4), it is easy to see that the following Bott-Chern cohomology classes

$$[\eta^{1\bar{1}}]_{BC},\quad [\eta^{3\bar{3}}]_{BC},\quad [\eta^3]_{BC},$$

are non zero. Also, we have that

$$\eta^{1\bar{1}}\wedge\eta^{3\bar{3}}=\partial\bar{\partial}\left(-\frac{1}{A\bar{B}}\eta^{2\bar{4}}\right),\quad \eta^{3\bar{3}}\wedge\eta^3=0. \tag{6.4.5}$$

Then, it is well defined the following Aeppli-Bott-Chern-Massey product

$$\langle[\eta^{1\bar{1}}]_{BC},[\eta^{3\bar{3}}]_{BC},[\eta^3]_{BC}\rangle_{ABC}=\left[-\frac{1}{A\bar{B}}\eta^{23\bar{4}}\right]_A\in\frac{H^{2,1}_A(M)}{H^{1,0}_A(M)\cup[\eta^{1\bar{1}}]_{BC}+H^{1,1}_A(M)\cup[\eta^3]_{BC}}.$$

Since $d*\eta^{23\bar{4}}=-d(\eta^{123\bar{1}\bar{4}})=0$, the form $\eta^{23\bar{4}}$ is Aeppli harmonic, hence, as a cohomology class in $H^{2,1}_A(M)$, we have that

$$\left[-\frac{1}{A\bar{B}}\eta^{23\bar{4}}\right]_A\neq0.$$

It remains to show that $\left[-\frac{1}{A\overline{B}}\eta^{23\overline{4}}\right]_A \notin H^{1,0}_A(M) \cup [\eta^{1\overline{1}}]_{BC} + H^{1,1}_A(M) \cup [\eta^3]_{BC}$.

Let us then suppose, by contradiction, the opposite, i.e.,

$$-\frac{1}{A\overline{B}}\eta^{23\overline{4}} = \sum_{i=1}^{h^{1,0}_A} r_i \xi^i \wedge \eta^{1\overline{1}} + \sum_{j=1}^{h^{1,1}_A} s_j \psi^j \wedge \eta^3 + \partial R + \overline{\partial} S, \tag{6.4.6}$$

where $h^{p,q}_A := \dim \mathcal{H}^{p,q}_A(M,g)$, $r_i, s_j \in \mathbb{C}$, $R \in \mathcal{A}^{1,1}(M)$, $S \in \mathcal{A}^{2,0}(M)$, and $\{\xi^i\}$ and $\{\psi^j\}$ are the left-invariant harmonic representatives of, respectively, $H^{1,0}_A(M)$, and $H^{1,1}_A(M)$, with respect to g.

It is immediate to compute the invariant Aeppli cohomology of (M,J) of bi-degree $(1,0)$ and $(1,1)$, resulting in

$$\xi^1 = \eta^1, \quad \xi^2 = \eta^2, \quad \xi^3 = \eta^3, \quad \xi^4 = \eta^4,$$
$$\psi^1 = \eta^{1\overline{2}},\ \psi^2 = \eta^{1\overline{3}},\ \psi^3 = \eta^{1\overline{4}},\ \psi^4 = \eta^{2\overline{1}},\ \psi^5 = \eta^{2\overline{2}},\ \psi^6 = \eta^{2\overline{3}},\ \psi^7 = \eta^{3\overline{1}},$$
$$\psi^8 = \eta^{3\overline{2}},\ \psi^9 = \eta^{3\overline{3}},\ \psi^{10} = \eta^{3\overline{4}},\ \psi^{11} = \eta^{4\overline{1}},\ \psi^{12} = \eta^{4\overline{3}},\ \psi^{13} = \eta^{2\overline{4}} - \frac{A\overline{B}}{\overline{A}B}\eta^{4\overline{2}}.$$

Then, equation (6.4.6) can be rewritten as

$$\begin{aligned} -\frac{1}{A\overline{B}}\eta^{23\overline{4}} &= -r_2\eta^{12\overline{1}} - r_3\eta^{13\overline{1}} - r_4\eta^{14\overline{1}} - s_1\eta^{13\overline{2}} - s_2\eta^{13\overline{3}} - s_3\eta^{13\overline{4}} - s_4\eta^{23\overline{1}} - s_5\eta^{23\overline{2}} \\ &\quad - s_6\eta^{23\overline{3}} + s_{10}\eta^{34\overline{1}} + s_{11}\eta^{34\overline{3}} + s_{12}\eta^{34\overline{4}} - s_{13}\eta^{23\overline{4}} - s_{13}\frac{A\overline{B}}{\overline{A}B}\eta^{34\overline{2}} + \partial R + \overline{\partial} S. \end{aligned} \tag{6.4.7}$$

We note that the form $\eta^{12\overline{134}}$ is d-closed. Therefore, if we multiply (6.4.7) by $\eta^{12\overline{134}}$, we obtain

$$0 = s_{13}\frac{A\overline{B}}{\overline{A}B}\eta^{1234\overline{1234}} + \partial(R \wedge \eta^{12\overline{134}}) + \overline{\partial}(S \wedge \eta^{12\overline{134}}),$$

i.e.,

$$s_{13}\frac{A\overline{B}}{\overline{A}B}\mathrm{Vol} = \partial(-R \wedge \eta^{12\overline{134}}) + \overline{\partial}(-S \wedge \eta^{12\overline{134}}). \tag{6.4.8}$$

By integrating (6.4.8) and applying Stokes theorem on a manifold with empty boundary, we obtain that $s_{13} = 0$.

If we repeat the same argument, multiplying now (6.4.7) by the d-closed form $\eta^{14\overline{123}}$, we obtain

$$\frac{1}{A\overline{B}}\mathrm{Vol} = \partial(R \wedge \eta^{14\overline{123}}) + \overline{\partial}(S \wedge \eta^{14\overline{123}}),$$

which, by integrating and Stokes theorem, leads to a contradiction.

To summarize,

$$\langle [\eta^{1\overline{1}}]_{BC}, [\eta^{3\overline{3}}]_{BC}, [\eta^3]_{BC} \rangle_{ABC} \neq 0,$$

i.e., we obtained a non vanishing Aeppli-Bott-Chern-Massey product, which, by [18, Theorem 2.4], implies that (M,J) does not admit geometrically-Bott-Chern-formal metrics.

(ii) **The product of two Inoue surfaces of type $\mathcal{S}_M$.**

Let $(M,J) = (\mathcal{S}_M, J_{\mathcal{S}_M}) \times (\mathcal{S}_M, J_{\mathcal{S}_M})$ be product of two Inoue surfaces of type $\mathcal{S}_M$. The complex structure J is determined by the left-invariant $(1,0)$-coframe $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ with structure equations

$$\begin{cases} d\eta^1 = \frac{\alpha - i\beta}{2i}\eta^{12} - \frac{\alpha - i\beta}{2i}\eta^{1\overline{2}} \\ d\eta^2 = -i\alpha\eta^{2\overline{2}} \\ d\eta^3 = \frac{\gamma - i\delta}{2i}\eta^{34} - \frac{\gamma - i\delta}{2i} - \frac{\gamma - i\delta}{2i}\eta^{3\overline{4}} \\ d\eta^4 = -i\gamma\eta^{4\overline{4}}, \end{cases} \tag{6.4.9}$$

for $\alpha, \gamma \in \mathbb{R} \setminus \{0\}$, $\beta, \delta \in \mathbb{R}$.

From (6.4.9), it is clear that the following Bott-Chern cohomology classes

$$[\eta^{2\bar{2}}]_{BC}, \quad [\eta^{34\bar{3}}]_{BC}, \quad [\eta^{4\bar{4}}]_{BC}$$

are well defined and non zero. Moreover,

$$\eta^{2\bar{2}} \wedge \eta^{34\bar{3}} = \partial\bar{\partial}(-\frac{1}{2\alpha\gamma})\eta^{23\bar{3}}, \qquad \eta^{34\bar{3}} \wedge \eta^{4\bar{4}} = 0,$$

hence the following Aeppli-Bott-Chern-Massey product

$$\langle [\eta^{2\bar{2}}_{BC}, [\eta^{34\bar{3}}]_{BC}, [\eta^{4\bar{4}}]_{BC} \rangle_{ABC} = \left[\frac{1}{\alpha\gamma}\eta^{234\bar{3}\bar{4}}\right]_A \in \frac{H^{3,2}_A(M)}{H^{2,1}_A \cup [\eta^{2\bar{2}}]_{BC} + H^{2,1}_A(M) \cup [\eta^{4\bar{4}}]_{BC}},$$

is well defined.

Note that since $d(*_g \eta^{234\bar{3}\bar{4}}) = -d(\eta^{12\bar{1}}) = 0$, the form $\eta^{234\bar{3}\bar{4}}$ is Aeppli-harmonic and, as a Aeppli cohomology class,

$$\left[\frac{1}{\alpha\gamma}\eta^{234\bar{3}\bar{4}}\right]_A \neq 0.$$

It remains to show that $\left[\frac{1}{\alpha\gamma}\eta^{234\bar{3}\bar{4}}\right]_A \notin H^{2,1}_A \cup [\eta^{2\bar{2}}]_{BC} + H^{2,1}_A(M) \cup [\eta^{4\bar{4}}]_{BC}$. In order to do, we prove that $H^{2,1}_A(M) = \{0\}$, yielding that $H^{2,1}_A \cup [\eta^{2\bar{2}}]_{BC} + H^{2,1}_A(M) \cup [\eta^{4\bar{4}}]_{BC} = 0$.

By definition, we observe that

$$H^{2,1}_A(M) := \frac{\operatorname{Ker}(\partial\bar{\partial}|_{\mathcal{A}^{2,1}(M)})}{\operatorname{Im}(\partial|_{\mathcal{A}^{1,1}(M)}) + \operatorname{Im}(\bar{\partial}|_{\mathcal{A}^{2,0}(M)})}$$

With the aid of structure equations (6.4.9) and Sagemath, we can compute

$$\begin{aligned}
&\dim_{\mathbb{C}} \operatorname{Ker}(\partial\bar{\partial}|_{\wedge^{2,1}\mathfrak{g}}) = 15\\
&\dim_{\mathbb{C}} \operatorname{Im}(\partial|_{\wedge^{1,1}\mathfrak{g}}) = 12\\
&\dim_{\mathbb{C}} \operatorname{Im}(\bar{\partial}|_{\wedge^{2,0}\mathfrak{g}}) = 6\\
&\dim_{\mathbb{C}} \operatorname{Im}(\partial|_{\wedge^{1,1}\mathfrak{g}}) \cap \operatorname{Im}(\bar{\partial}|_{\wedge^{2,0}\mathfrak{g}}) = 3,
\end{aligned}$$

so that

$$\begin{aligned}
\dim_{\mathbb{C}} H^{2,1}_A(M) &= \dim_{\mathbb{C}} \operatorname{Ker}(\partial\bar{\partial}|_{\wedge^{2,1}\mathfrak{g}}) - \dim_{\mathbb{C}}(\operatorname{Im}(\partial|_{\wedge^{1,1}\mathfrak{g}}) + \operatorname{Im}(\bar{\partial}|_{\wedge^{2,0}\mathfrak{g}}))\\
&= 15 - (18 - 3) = 0.
\end{aligned}$$

Therefore, $H^{2,1}_A(M) = \{0\}$ and

$$\langle [\eta^{2\bar{2}}]_{BC}, [\eta^{34\bar{3}}]_{BC}, [\eta^{4\bar{4}}]_{BC} \rangle_{ABC} = \left[\frac{1}{\alpha\gamma}\eta^{234\bar{3}\bar{4}}\right]_A \neq 0,$$

which, by [18, Theorem 2.4] implies that (M, J) does not admit any geometrically-Bott-Chern-formal metric.

(iii) **The product of a Inoue surface of type $\mathcal{S}_M$ and a primary Kodaira surface.** Let $(M, J) = (\mathcal{S}_M, J_{\mathcal{S}_M}) \times (KT, J_{KT})$ be the product of a Inoue surface of type $\mathcal{S}_M$ and a primary

Kodaira surfaces. The complex structure J is determined by the coframe of left-invariant $(1,0)$-form $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ with structure equations

$$\begin{cases} d\eta^1 = \frac{\alpha - i\beta}{2i}\eta^{12} - \frac{\alpha - i\beta}{2i}\eta^{1\overline{2}} \\ d\eta^2 = -i\alpha\eta^{2\overline{2}} \\ d\eta^3 = 0 \\ d\eta^4 = A\eta^{3\overline{3}}, \end{cases} \tag{6.4.10}$$

with $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{R}$, $B \in \mathbb{C} \setminus \{0\}$.

We consider the following Bott-Chern cohomology classes

$$[\eta^{2\overline{2}}]_{BC}, \quad [\eta^{3\overline{3}}]_{BC}, \quad [\eta^3]_{BC}.$$

They are clearly well defined and they are not zero. Moreover,

$$\eta^{2\overline{2}} \wedge \eta^{3\overline{3}} = \partial\overline{\partial}\left(\frac{1}{i\alpha\overline{A}}\eta^{2\overline{4}}\right), \quad \eta^{3\overline{3}} \wedge \eta^3 = 0.$$

Therefore, the Aeppli-Bott-Chern-Massey product

$$\langle [\eta^{2\overline{2}}]_{BC}, [\eta^{3\overline{3}}]_{BC}, [\eta^3]_{BC}\rangle_{ABC} = \left[\frac{1}{i\alpha\overline{A}}\eta^{23\overline{4}}\right]_A \in \frac{H^{2,1}_A(M)}{H^{1,0}_A(M) \cup [\eta^{2\overline{2}}]_{BC} + H^{1,1}_A(M) \cup [\eta^3]_{BC}}$$

is well defined.

Note that $d * (\frac{1}{i\alpha\overline{A}}\eta^{23\overline{4}}) = -\frac{1}{i\alpha\overline{A}}d(\eta^{123\overline{1}\overline{4}}) = 0$, i.e., the form $\frac{1}{i\alpha\overline{A}}\eta^{23\overline{4}}$ is Aeppli-harmonic and

$$\left[\frac{1}{i\alpha\overline{A}}\eta^{23\overline{4}}\right]_A \neq 0,$$

as a Aeppli cohomology class.

It remains to show that $\left[\frac{1}{i\alpha\overline{A}}\eta^{23\overline{4}}\right]_A \notin H^{1,0}_A(M) \cup [\eta^{2\overline{2}}]_{BC} + H^{1,1}_A(M) \cup [\eta^3]_{BC}$. Let us suppose by contradiction that this is the case, i.e.,

$$\frac{1}{i\alpha\overline{A}}\eta^{23\overline{4}} = \sum_{i=1}^{h^{1,0}_A} \lambda_i \xi^i \wedge \eta^{2\overline{2}} + \sum_{j=1}^{h^{1,1}_A} \mu_j \psi^j \wedge \eta^3 + \partial R + \overline{\partial} S, \tag{6.4.11}$$

with $\lambda_i, \mu_j \in \mathbb{C}$, $R \in \mathcal{A}^{1,1}(M)$, $S \in \mathcal{A}^{2,0}(M)$, and $\{\xi^i\}$ and $\{\mu^j\}$ are, respectively, a basis for $\mathcal{H}^{1,0}_A(M,g)$ and $\mathcal{H}^{1,1}_A(M,g)$. By structure equations (6.4.10), we can compute the spaces of Aeppli-harmonic forms with respect to g

$$\mathcal{H}^{1,0}_A(M,g) = \langle \eta^3 \rangle, \quad \mathcal{H}^{1,1}_A(M,g) = \langle \eta^{3\overline{3}}, \eta^{3\overline{4}}, \eta^{4\overline{3}} \rangle.$$

Then, equation (6.4.11) becomes

$$\frac{1}{i\alpha\overline{A}}\eta^{23\overline{4}} = -\lambda_1 \eta^{23\overline{3}} + \partial R + \overline{\partial} S. \tag{6.4.12}$$

Since the form $\eta^{14\overline{1}\overline{2}\overline{3}}$ is d-closed, if we multiply (6.4.12) by $\eta^{14\overline{1}\overline{2}\overline{3}}$, we obtain

$$\frac{1}{i\alpha\overline{A}}\mathrm{Vol} = \partial(-R \wedge \eta^{14\overline{1}\overline{2}\overline{3}}) + \overline{\partial}(-S \wedge \eta^{14\overline{1}\overline{2}\overline{3}}),$$

which, by integrating over M and applying Stokes theorem, leads to contradiction.

Hence, we showed that

$$\langle [\eta^{2\overline{2}}]_{BC}, [\eta^{3\overline{3}}]_{BC}, [\eta^3]_{BC}\rangle_{ABC} \neq 0,$$

i.e., (M,J) admits a non vanishing Aeppli-Bott-Chern-Massey product. By [18, Theorem 2.4], this implies that (M,J) does not admit any geometrically-Bott-Chern-formal metric. □

We prove one more result in this direction, showing that the existence of Aeppli-Bott-Chern-Massey products obstructs the existence of geometrically-Bott-Chern-formal metrics on a family of 4-dimensional complex nilmanifolds which cannot be constructed as a product of two or more manifolds.

We start by considering the set of complex forms $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ of type $(1,0)$ satisfying the following structure equations

$$\begin{cases} d\eta^1 = 0 \\ d\eta^2 = 0 \\ d\eta^3 = A\eta^{2\bar{1}} \\ d\eta^4 = B_1\eta^{12} + B_2\eta^{1\bar{1}} + B_3\eta^{2\bar{2}}, \end{cases}$$

with $A, B_1, B_2, B_3 \in \mathbb{Q}[i]$. Let $\mathfrak{g}^* = \mathrm{Span}_{\mathbb{R}}\langle \mathfrak{Re}(\eta^i)\mathfrak{Im}(\eta^i)\rangle_{i=1,\dots,4}$. Then, setting $\mathfrak{g}^{1,0} = \mathrm{Span}_{\mathbb{C}}\langle \eta^1, \eta^2, \eta^3, \eta^4\rangle$, we obtain that $\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}}$ gives rise to an integrable almost complex structure J on the real nilpotent Lie algebra $\mathfrak{g}$. We will consider the natural complex structure J on $\mathfrak{g}$ which arises by choosing $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ as a coframe of $(1,0)$-form on $\mathfrak{g}^*_{\mathbb{C}}$. Let G be the simply-connected and connected Lie group with Lie algebra $\mathfrak{g}$. By Malcev's theorem we have that the 8-dimensional real Lie group G associated to $\mathfrak{g}$ admits a discrete uniform subgroup Γ such that $M := \Gamma\backslash G$ is compact and, in particular, (M, J) is an 8-dimensional nilmanifold with an invariant complex structure.

Moreover, since J is nilpotent complex structure on (M, J), by [11, Theorem 3.8] (see also, [123, Corollary 3.12]), we have the following isomorphisms

$$H^{p,q}_{BC}(\mathfrak{g}, J) \to H^{p,q}_{BC}(M),$$

i.e., the Bott-Chern cohomology of (M, J) can be computed by means of the complex of left-invariant forms on $\mathfrak{g}$.

Theorem 6.4.6 ([133])**.** *Let $M = \Gamma\backslash G$ be a complex 4-dimensional nilmanifold endowed with the left-invariant complex structure J determined by a coframe of $(1,0)$-forms $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ with structure equations*

$$\begin{cases} d\eta^1 = 0 \\ d\eta^2 = 0 \\ d\eta^3 = A\eta^{2\bar{1}} \\ d\eta^4 = B_1\eta^{12} + B_2\eta^{1\bar{1}} + B_3\eta^{2\bar{2}}, \end{cases} \tag{6.4.13}$$

with $A \in \mathbb{C} \setminus \{0\}$, $B_j \in \mathbb{Q}[i]$, such that

$$|A|^2 + |B_2|^2 = 2\mathfrak{Re}\,(B_2\overline{B}_3). \tag{6.4.14}$$

Then (M, J) admits a SKT metric but does not admit any geometrically-Bott-Chern-formal metric.

Proof. Let now consider the diagonal metric g with fundamental associated form

$$F = \frac{i}{2}\sum_{h=1}^{4} \eta^{h\bar{h}}.$$

With the aid of (6.4.13) and (6.4.14), we can see clearly that g is SKT, i.e., $\partial\overline{\partial}F = 0$.

We will show that (M, J) admits a non vanishing ABC-Massey product, which suffices to prove that there exists no geometrically-Bott-Chern-formal metric on (M, J).

Let us consider the following Bott-Chern cohomology classes

$$[\eta^{1\bar{1}}]_{BC}, \quad [\eta^{2\bar{2}}]_{BC}, \quad [\eta^{2}]_{BC}.$$

Since

$$\eta^{2\bar{2}} \wedge \eta^2 = 0, \quad \eta^{1\bar{1}} \wedge \eta^{2\bar{2}} = \partial\bar{\partial}(\frac{1}{|A|^2}\eta^{3\bar{3}}),$$

the Aeppli-Bott-Chern-Massey product

$$\langle [\eta^{1\bar{1}}]_{BC}, [\eta^{2\bar{2}}]_{BC}, [\eta^{2}]_{BC} \rangle_{ABC} = \left[-\frac{1}{|A|^2}\eta^{23\bar{3}}\right]_A \in \frac{H_A^{2,1}(M)}{H_A^{1,0}(M) \cup [\eta^{1\bar{1}}]_{BC} + H_A^{1,1}(M) \cup [\eta^2]_{BC}},$$

is well defined.

We notice that the $d *_g (\frac{1}{|A|^2}\eta^{23\bar{3}}) = \frac{1}{|A|^2} d(\eta^{124\bar{1}\bar{4}}) = 0$, i.e., the form $\eta^{23\bar{3}}$ is Aeppli harmonic and, as a Aeppli cohomology class, we have that $[-\frac{1}{|A|^2}\eta^{23\bar{3}}]_A \neq 0$. It remains to show that $[-\frac{1}{|A|^2}\eta^{23\bar{3}}]_A \notin H_A^{1,0}(M) \cup [\eta^{1\bar{1}}]_{BC} + H_A^{1,1}(M) \cup [\eta^2]_{BC}$.

Let us now suppose by contradiction that $[-\frac{1}{|A|^2}\eta^{23\bar{3}}]_A \in H_A^{1,0}(M) \cup [\eta^{1\bar{1}}]_{BC} + H_A^{1,1}(M) \cup [\eta^2]_{BC}$. By straightforward computations, it is easy to check that the spaces $\mathcal{H}_A^{1,0}(M,g)$ and $\mathcal{H}_A^{1,1}(M,g)$ are generated, respectively, by $\langle \psi^j \rangle_{j=1}^2$ and $\langle \xi^i \rangle_{i=1}^{11}$, where

$$\psi^1 = \eta^1, \quad \psi^2 = \eta^2,$$

and

$$\begin{aligned} &\xi^1 = \eta^{1\bar{3}}, \quad \xi^2 = \eta^{1\bar{4}}, \quad \xi^3 = \eta^{2\bar{3}}, \quad \xi^4 = \eta^{2\bar{4}}, \\ &\xi^5 = \eta^{3\bar{1}}, \quad \xi^6 = \eta^{3\bar{2}}, \quad \xi^7 = \eta^{3\bar{4}}, \quad \xi^8 = \eta^{4\bar{1}}, \\ &\xi^9 = \eta^{4\bar{2}}, \quad \xi^{10} = \eta^{4\bar{3}}, \quad \xi^{11} = \eta^{3\bar{3}} + \eta^{4\bar{4}}. \end{aligned}$$

Then, $[-\frac{1}{|A|^2}\eta^{23\bar{3}}]_A \in H_A^{1,0}(M) \cup [\eta^{1\bar{1}}]_{BC} + H_A^{1,1}(M) \cup [\eta^2]_{BC}$ implies that

$$-\frac{1}{|A|^2}\eta^{23\bar{3}} = \sum_{i=1}^{2} r_i \psi^i \wedge \eta^{1\bar{1}} + \sum_{j=1}^{11} s_j \xi^j \wedge \eta^2 + \partial R + \bar{\partial} S, \tag{6.4.15}$$

for $r_i, s_j \in \mathbb{C}$, $R \in \mathcal{A}^{1,2}(M)$, $S \in \mathcal{A}^{2,1}(M)$, so that

$$\begin{aligned} -\frac{1}{|A|^2}\eta^{23\bar{3}} = & -r_2\eta^{12\bar{1}} - s_1\eta^{21\bar{3}} - s_2\eta^{12\bar{4}} + s_5\eta^{23\bar{1}} + s_6\eta^{23\bar{2}} + s_7\eta^{23\bar{4}} \\ & + s_8\eta^{24\bar{1}} + s_9\eta^{24\bar{2}} + s_{10}\eta^{24\bar{3}} + s_{11}\eta^{23\bar{3}} + s_{11}\eta^{24\bar{4}} + \partial R + \bar{\partial} S. \end{aligned} \tag{6.4.16}$$

We note that the form $\eta^{13\bar{1}\bar{2}\bar{3}}$ is d-closed, therefore, if we multiply (6.4.16) by $\eta^{13\bar{1}\bar{2}\bar{3}}$, we obtain

$$0 = s_{11}\eta^{1234\bar{1}\bar{2}\bar{3}\bar{4}} + \partial(R \wedge \eta^{13\bar{1}\bar{2}\bar{3}}) + \bar{\partial}(S \wedge \eta^{13\bar{1}\bar{2}\bar{3}}),$$

which, by integrating and applying Stokes theorem, forces $s_{11} = 0$. Equation (6.4.16) reduces to

$$\begin{aligned} -\frac{1}{|A|^2}\eta^{23\bar{3}} = & -r_2\eta^{12\bar{1}} - s_1\eta^{21\bar{3}} - s_2\eta^{12\bar{4}} + s_5\eta^{23\bar{1}} + s_6\eta^{23\bar{2}} \\ & + s_7\eta^{23\bar{4}} + s_8\eta^{24\bar{1}} + s_9\eta^{24\bar{2}} + s_{10}\eta^{24\bar{3}} + \partial R + \bar{\partial} S. \end{aligned} \tag{6.4.17}$$

Now, the form $\eta^{14\overline{124}}$ is d-closed, so if we multiply (6.4.17) by $\eta^{14\overline{124}}$, we obtain

$$-\frac{1}{|A|^2}\mathrm{Vol} = \partial(R \wedge \eta^{14\overline{124}}) + \overline{\partial}(S \wedge \eta^{14\overline{124}}),$$

which, by integration and applying Stokes theorem, leads to contradiction.

Therefore, we showed that

$$\langle [\eta^{1\bar{1}}]_{BC}, [\eta^{2\bar{2}}]_{BC}, [\eta^{2}]_{BC} \rangle_{ABC} \neq 0,$$

i.e., (M, J) admits a non vanishing Aeppli-Bott-Chern-Massey product, which implies that (M, J) does not admit any geometrically-Bott-Chern-formal metric. $\square$

Appendix A

We present the notion of complex manifold as a differentiable manifold endowed with an equivalence class of holomorphic atlases.

Let M be a differentiable manifold of real dimension $2n$. A *holomorphic chart* on M (U, φ) is the datum of a open set $U \subset M$ and a map $\varphi: U \to \varphi(U) \subset \mathbb{C}^n$, which is a homeomorphism. In particular, for every point $p \in U$, we have holomorphic coordinates $\varphi(p) = (z_1(p), \dots, z_n(p))$. We will denote such coordinates as $(z_1, \dots, z_n)$. A *holomorphic atlas* on M is a collection of holomorphic charts on M $\{(U_i, \varphi_i)\}_{i\in I}$ such that

- $\bigcup_{i\in I} U_i = M$, i.e., the family $\{U_i\}_{i\in I}$ is a covering of M;
- on any $U_i \cap U_j \neq \varnothing$, the transition function $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1}: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a biholomorphism between open sets of $\mathbb{C}^n$.

Let $\mathfrak{U} = \{(U_i, \varphi_i)\}_{i\in I}$, $\mathfrak{V} = \{(V_j, \psi_j)\}_{j\in J}$ be two holomorphic atlases on M. We say that a holomorphic chart (U_i, φ_i) of $\mathfrak{U}$ is *compatible* with a holomorphic chart (V_j, ψ_j) of $\mathfrak{V}$ if, on $U_i \cap V_j \neq \varnothing$, the map $\varphi_i \circ \psi_j^{-1}: \psi_j(U_i \cap V_j) \to \varphi_i(U_i \cap V_j)$ is s holomorphic map between open subsets of $\mathbb{C}^n$. Moreover, two atlases of M $\mathfrak{U}$ and $\mathfrak{V}$ are *compatible* if every holomorphic charts of $\mathfrak{U}$ is compatible with every holomorphic chart of $\mathfrak{V}$.

Remark. Compatibility yields an equivalence relation on the set of holomorphic atlases.

Definition. A *complex manifold* M of *complex dimension* n is a differential manifold M of real dimension $2n$ endowed with an equivalence class of holomorphic atlases.

We will denote the complex dimension of M by $\dim_{\mathbb{C}} M$.

For the sake of simplicity, from now on, unless specified, we will call an equivalence class of holomorphic atlases simply a “holomorphic atlas".

Let M be a complex manifold and let $f: M \to \mathbb{C}$ be a continuous map from M to $\mathbb{C}$. We say that f is a *holomorphic function on* M if, for every holomorphic chart (U, φ) of M, the map $f \circ \varphi^{-1}: \varphi(U) \to \mathbb{C}$ is holomorphic.

Remark. Given two holomorphic charts (U, φ), (V, ψ) in the same holomorphic atlas of M, with $U \cap V \neq \varnothing$, if $f \circ \varphi^{-1}$ is holomorphic where defined, then $f \circ \psi^{-1} = f \circ (\varphi^{-1} \circ \varphi) \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1})$ is holomorphic where defined.

Let M and N be two complex manifolds of complex dimension, respectively, n and m and let $f: M \to N$ be a continuous s map. We say that f is *holomorphic* if for every holomorphic chart (U, φ) of M and every holomorphic chart (V, ψ) of N such that $f(U) \cap V \neq \varnothing$, the map

$$\psi \circ f \circ \varphi^{-1}: \varphi(U \cap f^{-1}(V))) \to \psi(V)$$

is a holomorphic map between opens of $\mathbb{C}^n$ and $\mathbb{C}^m$. In particular, f is said a *biholomorphism* if f is a holomorphic homeomorphism, and the manifolds M and N are said *biholomorphic*.

Let M and let E be a two differentiable manifolds with, respectively $\dim_{\mathbb{R}} M = n$ and $\dim_{\mathbb{R}} E = n + 2r$. A *complex vector bundle of rank* r over M is a smooth map $\pi: E \to M$ such that, for every $p \in M$, $E_p := \pi^{-1}(p)$, i.e., the fiber of E over p, has a structure of a r-dimensional complex vector space and M admits a covering $\{U_i\}_{i \in I}$ such that there exists diffeomorphisms $\psi_i: \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$, for every $i \in I$, with the following properties:

- $\pi|_{\pi^{-1}(U_i)} = \pi_1 \circ \psi_i$, where π_1 is the projection on the first component of each $U_i \times \mathbb{C}^r$, i.e., the following diagram is commutative

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\psi_i"] \arrow[d, "\pi|_{\pi^{-1}(U_i)}"'] & U_i \times \mathbb{C}^n \arrow[dl, "\pi_1"] \\
U_i &
\end{tikzcd}$$

- for every $p \in U_i$, the restriction $\psi_i|_{E_p}: E_p \to \{p\} \times \mathbb{C}^r$ is a $\mathbb{C}$-linear map.

For every $i, j \in I$ such that $U_i \cap U_j \neq \varnothing$, the *transition functions* $\psi_{ij} := \psi_j \circ \psi_i^{-1}: \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$ are smooth maps. More precisely, ψ_{ij} is a diffeomorphism of $(U_i \cap U_j) \times C^r$ and, for every $p \in U_i \cap U_j$, the restriction $\psi_{ij}(p)$ of ψ_{ij} to $\{p\} \times \mathbb{C}^r$ gives a $\mathbb{C}$-linear automorphism of $\mathbb{C}^r$ and $p \mapsto \psi_{ij}(p) \in GL(r; \mathbb{C})$ is a smooth map.

A *holomorphic vector bundle of rank* r over a complex manifold M of complex dimension n differs in the fact the one requires E to be a complex manifold of complex dimension $n + r$, the map $\pi: E \to M$ to be holomorphic, and the maps ψ_i and the induced maps $p \mapsto \psi_{ij}(p) \in GL(r, \mathbb{C})$ to be holomorphic.

Remark. A complex vector bundle $\pi: E \to M$ of rank r is also a differentiable vector bundle of rank $2r$, by considering the underlying differentiable structure on E and the structure of real vector space on each fiber E_p, $p \in M$. Viceversa, a differentiable bundle $E \to M$ or rank r can be considered as complex vector bundle $E \otimes \mathbb{C} \to M$ or rank r by considering the complexification of each fiber $(E \otimes \mathbb{C})_p = E_p \otimes \mathbb{C}$. Moreover, a holomorphic vector bundle is clearly also a complex vector bundle, whereas a complex vector bundle over a complex manifold does not have in general a structure of holomorphic vector bundle.

As for differentiable and complex vector bundles, a holomorphic vector bundle E is uniquely determined by a *cocyle*, i.e., a covering $\{U_i\}_{i \in I}$ of the base manifold and holomorphic maps $\{\theta_{ij}: U_i \cap U_j \to GL(r; \mathbb{C})\}_{i,j \in I}$, where r is the rank of E, which satisfy the following properties:

- $\theta_{ii}(p) = \mathrm{id}_{\mathbb{C}^r}$, for every $p \in U_i$,
- $\theta_{ij}^{-1}(p) = \theta_{ji}(p)$, for every $p \in U_i \cap U_j$,
- $\theta_{ki}(p) \circ \theta_{jk}(p) \circ \theta_{ij}(p) = \mathrm{id}_{\mathbb{C}^r}$, for every $p \in U_i \cap U_j \cap U_k$.

Example (Holomorphic tangent bundle)**.** Let M be a complex manifold with $\dim_{\mathbb{C}} M = n$ and let $\{(U_i, \varphi_i)\}$ be a holomorphic atlas of M, with $\varphi_i = (z_1^i, \ldots, z_n^i)$. Locally, we consider the following sets

$$U_i \times \mathbb{C}^n = \{(p, v^i) |\ p \in U_i,\ v^i = {}^t(v_1^i, \ldots, v_n^i) \in \mathbb{C}^n\}.$$

On the intersections, if $p \in U_i \cap U_j \neq \varnothing$ and $(p, v^i) \in U_i \times \mathbb{C}^n$, $(p, v^j) \in U_j \times \mathbb{C}^n$, we have the relations

$$v_l^i = \sum_{k=1}^{n} \frac{\partial z_l^i}{\partial z_k^j} v_k^j, \quad \text{for} \quad l = 1, \ldots, n,$$

so that, if we set

$$\psi_{ij} := \left(\frac{\partial z_l^i}{\partial z_k^j}\right)_{l,k=1}^n \in GL(n;\mathbb{C}), \tag{6.4.18}$$

we have that $v^i = \psi_{ij} v^j$.

Then $\mathcal{TM} := \bigcup_i (U_i \times C^n)$, endowed with the identifications given by ψ_{ij} as in (6.4.18), is the *holomorphic tangent bundle of* M.

Note that the tangent bundle TM of the differentiable manifold underlying a complex manifold M of complex dimension n is a differentiable vector bundle of rank $2n$ and, by considering its complexified version $T_{\mathbb{C}}M := TM \otimes \mathbb{C}$, it is a complex vector bundle of rank $2n$ over M.

Let now $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be two holomorphic (respectively, complex) vector bundles over M of rank, respectively r_1, and r_2. Then a *map of holomorphic*, respectively *complex vector bundles* is a holomorphic, respectively differentiable map $\Phi : E_1 \to E_2$ such that the following diagram is commutative

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M & \xrightarrow{\mathrm{id}} & M \end{array}$$

and for every point $p \in M$, the restriction $\Phi(p) : (E_1)_p \to (E_2)_p$ is linear and the rank of $\Phi(p)$ is independent of $p \in M$.

Let now $f : M \to \mathbb{C}$ be a complex valued function on M. It may be useful to describe the local action of the exterior differential d on f. If $(z_1, \ldots, z_n)$ are holomorphic coordinates on a neighborhood $U \subset M$, then locally on U,

$$df = \partial f + \overline{\partial} f = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz^j + \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}^j} d\overline{z}^j,$$

where dz^j and $d\overline{z}^j$ are, respectively, the holomorphic $(1,0)$-forms and anti-holomorphic $(0,1)$-forms, dual to the $(1,0)$-vector fields $\frac{\partial}{\partial z^j}$ and $(0,1)$-vector fields $\frac{\partial}{\partial \overline{z}^j}$ induced by the holomorphic coordinates $(z^1, \ldots, z^n)$. In particular, by identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, the holomorphic coordinates can be thought of as $z^j = x^j + iy^j$ and one obtains that

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right), \qquad \frac{\partial}{\partial \overline{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right),$$

and

$$dz^j = dx^j + i dy^j, \qquad d\overline{z}^j = dx^j - i dy^j.$$

Note that with this notation, a complex function f is holomorphic if, and only if, $\overline{\partial} f = 0$.

Analougously, for any given form α on M locally written as

$$\alpha = \alpha_{I\overline{J}} dz^I \wedge d\overline{z}^J,$$

where $I = \{1 \le i_1 < \cdots < i_p \le n\}$ and $J = \{1 \le j_1 < \cdots < j_q \le n\}$ and $dz^I := dz^{i_1} \wedge \cdots \wedge dz^{i_p}$ and $d\overline{z}^J := d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}$, the local expression of the action of d is given by

$$d\alpha = \partial\alpha + \overline{\partial}\alpha = \sum_{k=1}^n \frac{\partial \alpha_{I\overline{J}}}{\partial z_k} dz^k \wedge dz^I \wedge d\overline{z}^J + \sum_{k=1}^n \frac{\partial \alpha_{I\overline{J}}}{\partial \overline{z}^k} d\overline{z}^k \wedge dz^I \wedge d\overline{z}^J.$$

We also recall the implicit formulas for the exterior differential on any differential form α on M. More specifically, if $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$ is any complex k-form (see Section 1) on M and $Z_1, \ldots, Z_{k+1} \in T_{\mathbb{C}}M$, then

$$d\alpha(Z_1, \ldots, Z_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} Z_j(\alpha(Z_1, \ldots, \hat{Z}_j, \ldots, Z_{k+1})$$
$$+ \sum_{1 \leq j < l \leq k+1} (-1)^{j+l} \alpha([Z_j, Z_l], Z_1, \ldots, \hat{Z}_j, \ldots, \hat{Z}_l, \ldots, Z_{k+1}).$$

Appendix B

In this appendix, we recall the notion of *formality according to Sullivan* and *geometric formality according to Kotschick* as introduced in [143], respectively, [87]. In order to do so, we lend the terminology from the category of differential graded algebras.

Let M be a n-dimensional smooth manifold and let $(\bigwedge^\bullet(M), d)$ be its de Rham complex. Such a complex has a structure of a *differential graded algebra* (shortly, *DGA*), i.e., the structure of an algebra $\mathcal{A}$ over some field $\mathbb{K}$ which is decomposable as $\mathcal{A} = \bigoplus_j \mathcal{A}_j$, where each $\mathcal{A}_j$ is a subalgebra of $\mathcal{A}$, and which is endowed with a differential $d_{\mathcal{A}}$, i.e., a $\mathbb{K}$-linear map $d\colon \mathcal{A} \to \mathcal{A}$ such that

- $d(\mathcal{A}_j) \subset \mathcal{A}_{j+1}$,
- $d(\alpha \cdot \beta) = d\alpha \cdot \beta + (-1)^{\deg \alpha} \alpha \cdot d\beta$, for every $\alpha, \beta \in \mathcal{A}$,
- $d^2 = 0$.

In this category, a morphism of DGA's is an algebras morphism $f\colon (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ such that $f(\mathcal{A}_j) \subset \mathcal{B}_j$ and $d_{\mathcal{B}} \circ f = f \circ d_{\mathcal{A}}$. The *cohomology of a DGA* $(\mathcal{A}, d_{\mathcal{A}})$ is the DGA $(H^\bullet_{\mathcal{A}}, 0)$, where each space $H^k_{\mathcal{A}}$ is defined as

$$H^k_{\mathcal{A}} \coloneqq \frac{\operatorname{Ker}(d_{\mathcal{A}}\colon \mathcal{A}_j \to \mathcal{A}_{j+1})}{\operatorname{Im}(d_{\mathcal{A}}\colon \mathcal{A}_{j-1} \to \mathcal{A}_j)}.$$

By definition, a morphism of DGA's $f\colon (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ commutes with the differentials $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$, hence it induces a DGA's morphism at the level of cohomologies

$$H_f\colon (H^\bullet_{\mathcal{A}}, 0) \to (H^\bullet_{\mathcal{B}}, 0)$$

by $H_f([\alpha]) \coloneqq [f(\alpha)]$, for every $[\alpha] \in H^\bullet_{\mathcal{A}}$. We say that a morphism of DGA's $f\colon (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ is a *quasi-isomorphism* if H_f is an isomorphism.

Let us now consider two DGA's $(\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{B}, d_{\mathcal{B}})$. We say that $(\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{B}, d_{\mathcal{B}})$ are *equivalent as DGA's* if there exists a family of DGA's $\{(\mathcal{C}_j, d_{\mathcal{C}_j})\}_{j=1}^{2k+1}$ such that $(\mathcal{C}_0, d_{\mathcal{C}_0}) = (\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{C}_{2k+1}, d_{\mathcal{C}_{2k+1}}) = (\mathcal{B}, d_{\mathcal{B}})$, for each $i \in \{1, \dots, k\}$, there exists morphisms f_i and g_i

$$\begin{tikzcd} & (\mathcal{C}_{2i}, d_{\mathcal{C}_{2i}}) \arrow[dl, "f_i"'] \arrow[dr, "g_i"] & \\ (\mathcal{C}_{2i-1}, d_{\mathcal{C}_{2i-1}}) & & (\mathcal{C}_{2i+1}, d_{\mathcal{C}_{2i+1}}) \end{tikzcd}$$

such that H_{f_i} and H_{g_i} are quasi-isomorphisms, for every $i \in \{1, \dots, k\}$.

Definition. A DGA $(\mathcal{A}, d_{\mathcal{A}})$ is said *formal* if it is equivalent, as a DGA, to a DGA $(\mathcal{B}, d_{\mathcal{B}})$ whose differential $d_{\mathcal{B}}$ is identically zero, i.e., $d_{\mathcal{B}} \equiv 0$.

Whence, the definition of formality according to Sullivan, see [143, Definition].

Definition. A differentiable manifold M is said to be *formal according to Sullivan* if its de Rham complex $(\mathcal{A}^\bullet(M), d)$ is a formal DGA.

Examples of formal manifolds are compact Kähler manifold, or more generally, complex manifolds satisfying the $\partial\overline{\partial}$-lemma, see, e.g., [47]. However, in general a manifold is not formal according to Sullivan; more precisely, there exist certain cohomological obstructions. We present here the definition adapted to the de Rham complex of a differentiable manifold M, see [99].

Definition. Let $[\alpha] \in H^p_{dR}(M)$, $[\beta] \in H^q_{dR}(M)$, and $[\gamma] \in H^r_{dR}(M)$ be de Rham cohomology classes on M such that $[\alpha] \cup [\beta] = 0 \in H^{p+q}_{dR}(M)$ and $[\beta] \cup [\gamma] = 0 \in H^{q+r}_{dR}(M)$, i.e., there exist $f_{\alpha\beta} \in \mathcal{A}^{p+q-1}(M)$ and $f_{\beta\gamma} \in \mathcal{A}^{q+r-1}(M)$ such that

$$\alpha \wedge \beta = df_{\alpha\beta}, \qquad \beta \wedge \gamma = df_{\beta\gamma}.$$

Then, the *Massey product* $\langle [\alpha], [\beta], [\gamma] \rangle$ is the equivalence class of de Rham cohomology classes defined as

$$\langle [\alpha], [\beta], [\gamma] \rangle := [f_{\alpha\beta} \wedge \gamma - (-1)^p \alpha \wedge f_{\beta\gamma}] + \mathcal{J} \in \frac{H^{p+q+r-1}_{dR}(M)}{\mathcal{J}},$$

where $\mathcal{J}$ is the ideal of $H^{p+q+r-1}_{dR}(M)$ defined by $\mathcal{J} := [\alpha] \cup H^{q+r-1}_{dR}(M) + [\gamma] \cup H^{p+q-1}_{dR}(M)$.

Massey products do not depend on the representatives α, β, and γ nor on the primitives $f_{\alpha\beta}$ and $f_{\beta\gamma}$, hence they are well defined. Moreover, if $f\colon (\mathcal{A}, d_{\mathcal{A}}) \to (\mathcal{B}, d_{\mathcal{B}})$ is a DGA's morphism, then it is easy to see that Massey products are compatible with H_f, i.e., it holds

$$H_f \langle [\alpha], [\beta], [\gamma] \rangle = \langle H_f[\alpha], H(f)[\beta], H(f)[\gamma] \rangle.$$

Hence, we immediately have the following.

Proposition. *On a formal manifold M, all Massey products vanish.*

In general, on a compact differentiable manifold M, the choice of representatives for the de Rham cohomology yields just a structure of A_∞-algebra in the sense of Stasheff [134], by the Homotopy Transfer Principle by Kadeishvili [80], see *e.g.* [156, 102]. We refer to [97, 34] for understanding the relationship between the higher multiplications and the Massey products. Such an A_∞-algebra is actually an algebra if and only if X is formal according to Sullivan. In particular, when we can choose a specific set of representatives, we obtain a strong notion of formality, which has then been introduced by Kotschick in [87]. Let M be a differentiable manifold endowed with a Riemannian metric g. Then, set $d^* := - * \circ d \circ *$, where $*$ is the usual Hodge operator on $\mathcal{A}^\bullet(M)$, and set $\Delta := dd^* + d^*d$, i.e., the usual Hodge Laplacian, and

$$\mathcal{H}^k_\Delta(M) := \{\alpha \in \mathcal{A}^k(M) : \Delta\alpha = 0\}$$

the *harmonic k-forms on M* with respect to g, for every $k \in \{1, \dots, \dim M\}$. Recall that, by Hodge theory, the following isomorphisms of real vector spaces hold

$$\mathcal{H}^k_\Delta(M) \to H^k_{dR}(M).$$

In fact, whereas de Rham cohomology has a structure of algebra induced by the $\cup$ product of cohomology classes, the wedge product in general does not induce a structure of algebra on the space of harmonic forms. Hence, the following definition.

Definition. The Riemannian metric g is said to be *formal according to Kotschick* if $\mathcal{H}^\bullet_\Delta(M)$ has a structure of algebra induced by the $\wedge$ product. A differentiable manifold admitting a formal metric according to Kotschick is said to be *geometrically formal*.

As observed by Kotschick, globally symmetric spaces are geometrically formal and any Riemannian metric on rational homology spheres is formal according to Kotschick. Moreover, examples can be constructed by taking products of formal manifolds. For geometrically formal manifolds, as Sullivan points out in [143], the following holds.

Proposition. *Geometrically formal manifolds are formal according to Sullivan.*

Appendix C

In this appendix, the main facts about the geometry of Lie groups are recalled.

Let G be a connected differential manifold and let

$$\begin{aligned} \cdot : G \times G &\to G \\ (g,h) &\mapsto g \cdot h \end{aligned}$$

be a group operation on G, i.e., $\cdot$ is associative, admits an identity $e \in G$ and every $g \in G$ admits an inverse $g^{-1} \in G$ with respect to $\cdot$. If the operations $(g,h) \mapsto g \cdot h$ and $g \mapsto g^{-1}$ are $\mathcal{C}^{\infty}$ maps with respect to the differentiable structure on G, then $(G, \cdot)$ is a said a *Lie group.*

If $(G_1, \cdot_1)$ and $(G_2, \cdot_2)$ are two Lie groups, a group homomorphism $\varphi: G_1 \to G_2$ is said to be a *homomorphism of Lie groups* if φ is also a $\mathcal{C}^{\infty}$ map of differentiable manifolds. An *isomorphism of Lie groups* is an invertible homomorphism of Lie groups φ such that φ^{-1} is a $\mathcal{C}^{\infty}$ map.

For every $g \in G$, the *right translation by* g (respectively, *left translation by* g) are the maps

$$\begin{aligned} R_g : G &\longrightarrow G \\ h &\mapsto R_g(h) := h \cdot g \end{aligned}$$

respectively,

$$\begin{aligned} L_g : G &\longrightarrow G \\ h &\mapsto L_g(h) := g \cdot h \end{aligned}$$

By definition of Lie group, both R_g and L_g are diffeomorphisms of G but they are not homomorphisms of Lie groups. Note that their differentials at a point $h \in G$ are

$$\begin{aligned} (dR_g)_h : T_h G &\to T_{hg} G \\ X_h &\mapsto (dR_g)_h(X_h) \end{aligned}$$

and

$$\begin{aligned} (dL_g)_h : T_h G &\to T_{gh} G \\ X_h &\mapsto (dL_g)_h(X_h). \end{aligned}$$

A $\mathcal{C}^{\infty}$ vector field $X \in \Gamma(G, TG)$ on G is said to be *right-invariant* (respectively, *left-invariant*) if $dL_g(X) = X$ (respectively, $dR_g X = X$), i.e., for every $h \in H$,

$$(dL_g)_h(X_h) = X_{gh}$$

respectively,

$$(dR_g)_h(X_h) = X_{hg},$$

A $\mathcal{C}^\infty$ differential form $\alpha \in \Gamma(G, T^*G)$ is said to be *right-invariant* (respectively, *left-invariant*) if $R_g^*(\alpha) = \alpha$ (respectively, $L_g^*(\alpha) = \alpha$), where R_g^*, respectively L_g^*, is the pull back of forms by the map R_g, respectively, L_g.

If on the set of left-invariant vector fields $\mathfrak{g} := \{X \in \Gamma(G, TG) : dL_g X = X, \forall g \in G\}$ one considers the bracket of vector fields

$$[X, Y] = XY - YX, \qquad X, Y \in \mathfrak{g},$$

then the space $(\mathfrak{g}, [\cdot, \cdot])$ has a structure of a *Lie algebra*, i.e., the structure of a vector space endowed with a bilinear anticommutative binar operation. In particular, $\mathfrak{g}$ is called the *Lie algebra associated to* G. Viceversa, for every Lie algebra $\mathfrak{h}$, there exists a unique (up to isomorphism) Lie group H such that $\mathfrak{h}$ is the Lie algebra associated to H.

On a Lie group G, it can be easily seen that the map

$$\begin{aligned} \mathfrak{g} &\to T_eG \\ X &\mapsto X_e \end{aligned}$$

is an isomorphism of Lie algebras, i.e., a linear isomorphisms compatible with the brackets of each space, hence $\mathfrak{g}$ can be identified with T_eG and $\dim \mathfrak{g} = n$. Analogously, the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$, i.e., the space of left-invariant differential forms on G, can be identified with T_e^*G. Note that, for every $\alpha \in \mathfrak{g}^*$, $X \in \mathfrak{g}$, $\alpha(X)$ is a left-invariant function on G, which implies that $\alpha(X) \in \mathbb{R}$ is a constant.

The implicit formula for the exterior differential applied to a 1-form $\alpha \in \Gamma(G, T^*G)$

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

yields that, if $\alpha \in \mathfrak{g}^*$, $X, Y \in \mathfrak{g}^*$, then

$$d\alpha(X, Y) = -\alpha([X, Y]).$$

Let then $\{e_1, \dots, e_n\}$ be a basis for $\mathfrak{g}$ and suppose that

$$[e_i, e_j] = c_{ij}^k e_k,$$

with $c_{ij}^k \in \mathbb{R}$. Then, if $\{e^1, \dots, e^n\}$ is the base of $\mathfrak{g}^*$ dual to $\{e_1, \dots, e_n\}$, it is easy to see that

$$de^i(e_j, e_k) = -c_{jk}^i.$$

Therefore, the *structure constants* c_{ij}^k characterizing the Lie algebra $\mathfrak{g}$ of a Lie group G are determined by either the brackets of a fixed base $\{e_1, \dots, e_n\}$ of $\mathfrak{g}$ or the *structure equations* of its dual base $\{e^1, \dots, e^n\}$.

Let G be a Lie group and $\mathfrak{g}$ its associated Lie algebra. The *derived series* is the sequence

$$\mathfrak{g}^{(0)} := \mathfrak{g}, \qquad \mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}], \qquad \mathfrak{g}^{(k)} := [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}],$$

whereas the *lower central series* is defined as

$$\mathfrak{g}_{(0)} := \mathfrak{g}, \qquad \mathfrak{g}_{(1)} := [\mathfrak{g}, \mathfrak{g}], \qquad \mathfrak{g}_{(k)} := [\mathfrak{g}, \mathfrak{g}_{(k-1)}].$$

The Lie algebra $\mathfrak{g}$ is said to be *solvable* (respectively, *nilpotent*), if the derived series (respectively, the lower central series), reaches $\{0\}$, i.e., there exists a k' (respectively, k'') such that $g^{(k')} = \{0\}$, respectively, $\mathfrak{g}_{(k'')} = \{0\}$. Consequently, the Lie group G is said to be *solvable* (respectively *nilpotent*) if $\mathfrak{g}$ is *solvable* (respectively, *nilpotent*). Note that for every k, $\mathfrak{g}^{(k)} \subset \mathfrak{g}_{(k)}$, hence a nilpotent algebra is also solvable.

Definition. A *solvmanifold* M is a compact quotient $M = H\backslash G$ of a simply connected solvable Lie group G and a closed subgroup $H \leq G$.

A *nilmanifold* M is a compact quotient $\Gamma\backslash G$ of a simply connected nilpotent Lie group G by a discrete uniform subgroup $\Gamma \leq G$.

Remark. (i) By Mal'cev theorem, a simply connected nilpotent Lie group G admits a discrete uniform subgroup Γ (so that $\Gamma\backslash G$ is a nilmanifold) if, and only if, G admits a basis such that the constant structures are rational numbers, see [98].
(ii) Notice that on a solvmanifold (respectively, nilmanifold) $H\backslash G$, left-invariant tensors on G such as vector fields, differential forms, metrics, and endomorphisms, are in particular left-invariant with respect to any element of H, therefore they descend to the quotient $H\backslash G$.

We end this section by recalling two classical fundamental results regarding the de Rham cohomology of solvmanifolds and nilmanifolds.

Let us consider $(\wedge^\bullet \mathfrak{g}^*, d)$, i.e., the complex of left-invariant forms on Lie group G endowed with the exterior differential d. Let Γ be a discrete uniform subgroup of G, and let us consider $M := \Gamma\backslash G$. Then, the following result assures that the inclusion

$$\wedge^\bullet \mathfrak{g}^* \hookrightarrow \mathcal{A}^\bullet(M)$$

yields an isomorphism between the *de Rham cohomology of* $\mathfrak{g}$, namely

$$H^k(\mathfrak{g}^*) := \frac{\mathrm{Ker}(d\colon \wedge^k \mathfrak{g}^* \to \wedge^{k+1} \mathfrak{g}^*)}{\mathrm{Im}(d\colon \wedge^{k-1} \mathfrak{g}^* \to \wedge^k \mathfrak{g}^*)}$$

and the usual de Rham cohomology $H^\bullet_{dR}(M;\mathbb{R})$ of the solvmanifold M.

Theorem. *([108, Theorem 1]) Let $M = \Gamma\backslash G$ be a nilmanifold, with G a simply connected nilpotent Lie group and $\Gamma \subset G$ a discrete uniform subgroup. Then, the inclusion*

$$\wedge^\bullet \mathfrak{g}^* \hookrightarrow \mathcal{A}^\bullet(M)$$

induces the isomorphism

$$H^\bullet(\mathfrak{g}^*) \simeq H^\bullet_{dR}(M;\mathbb{R}).$$

More in general, let G be a *completely-solvable* Lie group, i.e., its Lie algebra $\mathfrak{g}$ is isomorphic to a subalgebra of the upper triangular matrices in $\mathfrak{gl}(m,\mathbb{R})$ for some m. Note that, in particular, a completely solvable Lie groups is solvable and a nilpotent Lie group is completely solvable.

Theorem. ([70, Corollary 4.2]) *Let G be a simply connected completely solvable Lie group with Lie algebra $\mathfrak{g}$. Let H be a discrete uniform subgroup of G and $M := H\backslash G$. Then the injection*

$$\wedge^\bullet \mathfrak{g} \hookrightarrow \mathcal{A}^\bullet(M)$$

induces an isomorphism

$$H^\bullet(\mathfrak{g}) \simeq H^\bullet(M;\mathbb{R}).$$

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